=> P-120

Ques (14) Recall that c_0 is the space of sequences cgs to 0 and that the norm is $\|x\| = \max_n |x(n)|$. Prove that the norm is differentiable at x iff $\exists$ a unique n s.t. $|x(n)| = \|x\|$

Soln.

Well defined

If $x(n) = 0 \ \forall n \in \mathbb{N}$ then $\|x\| = 0$
i.e. max. exist at all $|x(n)| = 0$

Let $x(l) \neq 0$ for some $l \in \mathbb{N}$
then $|x(l)| \neq 0$
choose $\epsilon > 0$ s.t. $|x(l)| \geq \epsilon$
then as $x(n) \to 0$
$\Rightarrow \exists \ N \in \mathbb{N}$ s.t. $|x(n)| < \epsilon \ \forall n \geq N$
So as $|x(l)| \geq \epsilon > |x(n)| \ \forall n \geq N$
$\Rightarrow$ sup lies in $\{x(1), x(2), \ldots, x(N)\}$ also $l \leq N$
As this set has finite elements so its max. will exist i.e. sup = max.
Hence $\|x\| = \max_n |x(n)|$ is well defined

Now to prove $x \to \|x\|$ is diff iff $\exists$ unique n s.t. $\|x\| = |x(n)|$
(i.e. max exist at unique position)

First suppose $\exists$ unique n_0 s.t. $\|x\| = |x(n_0)|$
T.P.:- $x \to \|x\|$ is diff.
As $\exists$ unique n_0 s.t. $|x(n_0)| = \|x\|$
So $x \neq 0$ (zero seq.)
So $|x(l)| < |x(n_0)| \quad \forall l \in \mathbb{N} \setminus \{n_0\}$

Now let $n_1 \in \mathbb{N}$ s.t. $|x(n_1)|$ is second largest term of $|x(l)|$, $l \in \mathbb{N}$
i.e. $|x(l)| \leq |x(n_1)| \quad \forall l \in \mathbb{N} \setminus \{n_1\}$

Here $|x(n_1)|$ may exist at unique point or not
But will exist.
Define $y(l) = x(l) \quad \forall l \in \mathbb{N} \setminus \{n_0\}$
$y(l) = 0 \qquad l = n_0$
So y is seq not having n_0^{th} term.
also $y(l) \to 0$ as $x(l) \to 0$
and $\|y\| = \max_{l \in \mathbb{N} \setminus \{n_0\}} |y(l)|$

So $\exists\ n_1 \in \mathbb{N}|_{\{n_0\}}$ s.t.

$\|y\| = |y(n_1)| = |x(n_1)|$

As $|y(l)| \le \|y\| = |y(n_1)| \quad \forall\ l \in \mathbb{N}|_{\{n_0\}}$

So $|x(l)| \le |x(n_1)| < |x(n_0)| \quad \forall\ l \in \mathbb{N}|_{\{n_0\}}$

Let $|x(n_0)| - |x(n_1)| > 0$

Also $|x(n_0)| - |x(l)| \ge d \quad \forall\ l \in \mathbb{N}|_{\{n_0\}}$

Let $\epsilon = \frac{d}{3}$

Now to show $\exists$ a bdd l.o. $A: \mathcal{G} \to \mathbb{R}$

s.t. $\lim\limits_{h \to 0} \dfrac{\|x+h\| - \|x\| - A\cdot h}{\|h\|} = 0 \qquad \text{--- (1)}$

So as $h \to 0$, so we can find limit for $\|h\| < \epsilon$

i.e. for $|h(l)| < \epsilon \quad \forall\ l \in \mathbb{N}$

~~$\|x+h\|$ at n^{th} position~~

$|(x+h)(n)| = |x(n) + h(n)| \ge |x(n)| - |h(n)| \qquad (|x+y| \ge ||x|-|y|| \ge |x|-|y|)$

Now $\|x+h\|$ at n_0^{th} position

$|(x+h)(n_0)| \ge |x(n_0)| - |h(n_0)|$

$\ge |x(n_0)| - \epsilon \qquad \to (|h(l)| < \epsilon \Rightarrow -h(l) > -\epsilon)$

$> |x(l)| + \epsilon \qquad \leftarrow \left(\because |x(n_0)| - |x(l)| \ge d \text{ as } d = 3\epsilon > 2\epsilon;\ |x(n_0)| - |x(l)| > 2\epsilon;\ |x(n_0)| - \epsilon > |x(l)| + \epsilon \right)$

$> |x(l)| + |h(l)|$

$\ge |x(l) + h(l)|$

$= |(x+h)(l)|$

$\Rightarrow |(x+h)(n_0)| > |(x+h)(l)| \quad \forall\ l \in \mathbb{N}|_{\{n_0\}}$

So we get max $|(x+h)(l)|$ at $l = n_0$

And if $x(n) > 0$

then take $A\cdot h = h(n)$

and if $x(n) < 0$

take $A\cdot h = -h(n)$

Here $A: C \to \mathbb{R}$ is a bdd l.o.

then $A(h_1+h_2) = (h_1+h_2)(n)$

$= h_1(n) + h_2(n)$

$= A\cdot h_1 + A\cdot h_2$

$A(\alpha h) = \alpha\, A(h)$

$= \alpha\, A\cdot h.$

Also $\|A\| = \sup_{h\neq 0} \frac{\|A\cdot h\|}{\|h\|} = \sup \frac{|h(n)|}{\|h\|}$

as $|h(n)| < \|h\|$

$\Rightarrow \|A\| \leq 1$

$\Rightarrow$ A is bdd l.o.

for $h(l) = \begin{cases} 1 & \text{if } l = n \\ 0 & \text{other} \end{cases}$

then $\|A\| = 1$

Same for $x(n) < 0$

(Also By thm-2. (P-116) If
If f is bdd in nbd of x and if a linear map A has the property in eq^n (1), then A is bdd linear map, in other words, A is Frechet derivative of f at x)

Now if $x(n) > 0$

then $x(n) + h(n) > 0 \to$ (As $x(n)$

So $\lim_{h\to 0} \frac{\| |x(n)+h(n)| - |x(n)| - h(n) \|}{\|h\|}$

$= \lim_{h\to 0} \frac{\| x(n) + h(n) - x(n) - h(n) \|}{\|h\|} = 0$

If $x(n) < 0$

then $x(n) + h(n) < 0$

As $|x(n)| - |x(l)| > d > \epsilon > h(n)$

$-x(n) - |x(l)| > h(n)$

$0 \geq -|x(l)| > x(n) + h(n)$

$x(n) + h(n) < 0$

So $\lim_{h\to 0} \frac{\|x+h\| - \|x\| - A\cdot h\|}{\|h\|} = 0$

$\Rightarrow x - \|x\|$ is diff.

Converse

T.P. ! $x \to \|x\|$ is diff.

~~Let then~~ $\max_n |x(n)|$ is unique

On contrary let $\max |x(m)|$ is not unique

And let it exists at atleast 2 points

let $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$

i.e. $\|x\| = |x(n_1)| = |x(n_2)|$

So either $x(n_1) = x(n_2)$ or $x(n_1) = -x(n_2)$

Take a sequence

$$h^n(l) = \begin{cases} \frac{1}{n} & , l = n_1 \\ 0 & , \text{others.} \end{cases}$$

Let $x(n_1) > 0$

As $x \to \|x\|$ is diff.

So $\exists$ A a bdd $l.o.$

So $A(e_i) = \alpha_i \in \mathbb{R}$, $e_i(l) = \begin{cases} 1 & , l = i \\ 0 & , \text{others} \end{cases}$

Let $x = (x_1, x_2 - - x_n - -)$

$$y_n = \sum_{j=1}^{n} x(j)\, e_j$$

$$\|x - y_n\| = \|(0, 0, - - - x_{n+1}, x_{n+2} - - -)\| \to 0$$

[$\because$ $x \in c_0$ $\exists n \in \mathbb{N}$ s.t. $|x(n)| < \epsilon$.

~~Defi then~~ e_i i

Define $A \cdot h = A(h(n))$

$$= A \cdot \left(\sum_{n=1}^{\infty} h(n)\, e_n(l) \right)$$

$$= \sum_{n=1}^{\infty} h(n)\, \alpha_n \qquad [\because \text{A is bdd \& hence cts})$$

So for h^n seq we choose above

$$\lim_{h^n \to 0} \frac{\| \|x + h^n\| - \|x\| - A \cdot h^n \|}{\|h^n\|} = 0 \quad \text{as } n \to \infty$$

$$= \frac{\| x(n_1) + \frac{1}{n} - x(n_1) - \frac{1}{n}\alpha_i \|}{\frac{1}{n}} \Rightarrow \quad \frac{1}{n} - \frac{1}{n}\alpha_{n_1} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow 1 - \alpha_{n_1} \to 0 \quad \text{as } n \to \infty$$

If $x(n_2) \geq 0$ then $\alpha_{n_2} = 1$ for $h^n(l) = \begin{cases} \frac{1}{n} & l = n_2 \\ 0 & \text{oth.} \end{cases}$

If $x(n_2) \leq 0$ then $\alpha_{n_2} = 1$ for some h^n

Now take seq $h^n(l) = \begin{cases} \frac{1}{n}, & l = n_1, n_2 \\ 0 & \text{, oth.} \end{cases}$

Then $\left\| \dfrac{\|x + h^n\| - \|x\| - A \cdot h_n}{\|h^n\|} \right\|$ here $A \cdot h^n = \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$\cancel{\left| \dfrac{x(n_1) + \frac{1}{n} + h^n(n_1) - x(n_1)}{h^n(n_1)} \right| = 1 \neq 0}$

$$\frac{\left| x(n_1) + \frac{1}{n} - x(n_1) - \frac{2}{n} \right|}{\frac{1}{n}} = 1 \neq 0$$

$\Rightarrow$ $x \to \|x\|$ is not diff. which is contradiction

$\Rightarrow$ ran $|x(n)|$ is unique

Same seq works if

$x(n_1)$, $x(n_2) \leq 0$

or $x(n_1)$, $x(n_2) \geq 0$

or $x(n_1) \geq 0$, $x(n_2) \leq 0$: $h^n(l) = \begin{cases} -\frac{1}{n} & l = n_1 \\ \frac{1}{n} & l = n_2 \\ 0 & \text{, other.} \end{cases}$

with these change $A \cdot h$ changes only and rest of the following follows directly.