Department of Mathematics
MPhil/PhD Coursework Examination
Topics in Analysis

Time: 3 hrs. (March 2019)

Attempt FIVE questions in all. All symbols carry their usual meaning. Max. Marks : 70

1. (a) State Banach's Fixed Point Theorem. Pick two hypotheses from the statement and show, with the help of examples, that these hypotheses cannot be weakened. (1+4)

   (b) Let $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous mapping satisfying $|f(s, t_1) - f(s, t_2)| \leq \lambda |t_1 - t_2|$ where λ is a constant depending only on f. Show that the initial-value problem $x'(s) = f(s, x(s)), x(0) = \beta$ has a unique solution in $C[0, b]$. Hence or other wise show that the equation $x'(s) = cos(x(s)g(s)), x(0) = \beta$ has a unique solution for every $g \in C[0, b]$. (5+4)

2. (a) Let H be a real Hilbert space, M a closed subspace of H and $f : M \to M$ a non expansive mapping. Let $\{y_n\}$ be a sequence in M satisfying $\lim_{n\to\infty} \|F(y_n) - y_n\| = 0$. If x_0 is a weak limit point of $\{y_n\}$ prove that x_0 must be a fixed point of F. (4)

   (b) State Browder's fixed point Theorem. Let $X = l^2$ with its usual norm and the mapping f be given by $f(x) = (1, x_1, x_2, x_3, \dots)$ where $x = (x_1, x_2, \dots) \in l^2$. Show that F has no fixed points and state, with justification, why Browder's Theorem does not yield a fixed point in this case? (1+4)

   (c) Is the function $f : (0, 1) \to L^\infty(0, 1)$ Bochner integrable when (a) $f(t) := \chi_{(0,t)}$ (b) $f(t) := tx_0$ where x_0 is a fixed element of $L^\infty(0, 1)$. Justify. (3+2)

3. (a) State the Stone Weierstrass Theorem. (1)

   (b) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{A}$ be the set of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ where $N \in \mathbb{N}, c_n \in \mathbb{C}, \theta \in \mathbb{R}$. Show that $\mathcal{A}$ is an algebra which separates points of $\mathbb{T}$ and vanishes at no point of $\mathbb{T}$. Further, show that $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, d\theta = 0$ for $f \in \mathcal{A}$ and deduce that there are continuous functions on $\mathbb{T}$ which are not in the uniform closure of $\mathcal{A}$. Is the Stone Weierstrass theorem applicable here? Give reasons. (7)

   (c) For a mapping $f : D \to X$ where X is a normed space and D an open subset of X, explain the difference between the statements (a) f' is continuous at x (b) $f'(x)$ is continuous. Prove that if $f'(x)$ exists, then it is continuous and differentiable. (6).

4. (a) Let $g \in C[0, 1], \alpha \in \mathbb{C}$ be fixed. Define $F : C[0, 1] \to C[0, 1]$ by $F(x) = \alpha g \cdot x, x \in C[0, 1]$ where $\cdot$ represents pointwise multiplication. Find the derivative of F. (4)

(b) Let $f : D \to Y$ be differentiable at $x \in D$ and Z be a normed linear space and $B : Y \to Z$ be a bounded linear operator. Prove that $B \circ f$ is Frechet differentiable at x and find $(B \circ f)'(x)$. (4)

(c) Let $f : H \to \mathbb{R}$ where H is a Hilbert space and $a \in H$ be fixed. Define $f(x) := \langle a, x \rangle^2, x \in H$. Show that f is differentiable and find its derivative. (6)

5. (a) Let $z_1 = 3+3i$, $z_2 = -2i$ and $z_3 = \infty$. Compute the spherical distance between (a) z_1, z_2, (b) z_1, z_3 and (c) $\frac{1}{z_1}, \frac{1}{z_3}$. Does the spherical distance define a metric on the extended complex plane? Justify. (3 + 2)

(b) Show that if f is a meromorphic function in a domain D then it is continuous in D with respect to the spherical distance. (4)

(c) What is meant by a C_0 sequence in a domain D? Illustrate with a non-trivial example. Further, show that a C_0 sequence of meromorphic functions has a limit function F. What can be said about the function F? Give statements only. (5) 5

6. (a) When is a family of meromorphic functions called normal? Give a non-trivial example of a normal family of meromorphic functions. (3)

(b) Show that a family of normal holomorphic functions which is locally uniformally bounded in a domain D is normal in D. State an analog of this result for a family of meromorphic functions. Check whether or not the family $\{f_\epsilon : 0 < \epsilon \leq 1\}$ where $f_\epsilon(z) = \frac{z}{(z+\epsilon)}$ is normal on the disc of radius 1 centered at 2? (4+1+3)

(c) Define spherical derivative at a point $z_0 \in D$ of a meromorphic function defined on a domain D. Illustrate this definition with a non-trivial example. State a characterisation for normality of a family of meromorphic maps in terms of the spherical derivatives of its members. (1+1+1)

7. (a) State Montel's Theorem for normality of a family of meromorphic functions. Further, state the analog of this result for holomorphic functions. Illustrate one of these results. (3)

(b) Let $\mathcal{F}$ be a normal family of holomorphic functions in a domain D satisfying $\min_{z \in \sigma} |f(z)| \leq M \ \forall f \in \mathcal{F}$ where σ is a bounded and closed subset of D and M is a positive number. Show that the family $\mathcal{F}$ is uniformly bounded on each bounded closed subset E of D. (4)

(c) Let $f_n(z) := 2^n z + 2^{2n} z^2$ and $g_n(z) = \frac{z}{(z - \frac{1}{2^n})}$. What can you deduce about the normality of the families $\{f_n\}$ and $\{g_n\}$ in the domain $D = \{z : |z| < 2\}$. Justify. (2+2)

(d) What is meant by a simply covered image domain (SCID) of a holomorphic function? Illustrate with an example. State a criterion for normality of holomorphic functions involving (SCID). (1+1+1)