OB 19/20

Ques : Show that the supremum norm on space $X = C[0,1]$ is not differentiable at any element x for which there are two or more points t in $[0,1]$ where $|x(t)| = \|x\|_X$

Soln. : Let us assume that the sup norm on $X = C[0,1]$ is fretchet differentiable at an element x of $C[0,1]$ for which there are two or more points t in $[0,1]$ say t_1, t_2 with $t_1 \neq t_2$.

then $\exists$ a bounded linear functional $A_x : X \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{|\|x+h\|_X - \|x\|_X - A_x(h)|}{\|h\|_X} = 0 \quad \text{--- (1)}$$

If (1) holds for $h \to 0$, then it also holds for any sequence $\langle a_n \rangle$ in $C[0,1]$ converging to 0. So we can replace h by a_n and we will have

$$\lim_{a_n \to 0} \frac{|\|x+a_n\|_X - \|x\|_X - A_x(a_n)|}{\|a_n\|_X} = 0$$

Take $\langle a_n \rangle = \langle k_n v \rangle$ where $\langle k_n v \rangle$ is sequence in $C[0,1]$ where $\langle k_n \rangle$ is a real sequence converging to 0 and v is a fixed element of $C[0,1]$.

$$v = v(t) = \frac{|t - t_2|}{\||t - t_2|\|_X}$$

So, we have

$$\lim_{k_n \to 0} \frac{|\|x+k_n v\|_X - \|x\|_X - A_x(k_n v)|}{\|k_n v\|_X} = 0$$

$$\Rightarrow \lim_{k_n \to 0} \left| \frac{\|x+k_n v\|_X - \|x\|_X - k_n A_x(v)}{k_n \cdot \|v\|_X} \right| = 0 \quad \text{where } \|k_n v\|_X = |k_n| \|v\|_X$$

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$$\Rightarrow \lim_{k_n \to 0} \left| \frac{\|x + k_n v\|_x - \|x\|_x}{k_n} - A_x(v) \right| = 0$$

$$\Rightarrow \lim_{k_n \to 0} \frac{\|x + k_n v\|_x - \|x\|_x - A_x(v)}{k_n} = 0$$

$$\Rightarrow \lim_{k_n \to 0} \frac{\|x + k_n v\|_x - \|x\|_x}{k_n} = A_x(v)$$

Case 1 $\quad x(t_1) \geq 0$

Consider $\|x + k_n v\|_x - \|x\|_x \geq x + k_n v(t_1) - |x(t_1)|$

$$= \cancel{x(t_1)} + k_n v(t_1) - \cancel{x(t_1)}$$

$$= k_n \frac{|t_1 - t_2|}{\alpha} \quad \text{where } \alpha = \||t_1 - t_2|\|$$

$$\frac{\|x + k_n v\|_x - \|x\|_x}{k_n} \geq \frac{|t_1 - t_2|}{\alpha} > 0$$

So we get

$$\liminf_{k_n \to 0^+} \frac{\|x + k_n v\|_x - \|x\|_x}{k_n} > 0 \quad \text{——} (2)$$

Consider, $\|x + k_n v\|_x - \|x\|_x \geq |x(t_2) + k_n v(t_2)| - |x(t_2)|$

$$= |x(t_2)| - |x(t_2)| = 0$$

So $$\limsup_{k_n \to 0^-} \frac{\|x + k_n v\|_x - \|x\|_x}{k_n} \leq 0 \quad \text{——} (3)$$

From (2) and (3) we get

$\lim_{k_n \to 0} \frac{\|x + k_n v\|_x - \|x\|_x}{k_n}$ does not exist.

So $A_x(v)$ is not defined which is a contradiction to existence of A_x on $C[0,1]$ as bounded linear functional hence our assumption is wrong, supnorm on $C[0,1]$ is not frechet differentiable at an element x of $C[0,1]$ for which there are 2 or more points t in $[0,1]$ s.t $\|x\| = |x(t)|$

Case 2 $x(t_1) < 0$, Take $v = v(t) = -|t - t_2| / \| 1 - |t - t_2| \|$

Consider, $\|x + k_n v\|_X - \|x\|_X \geq (x - k_n v)(t_1) - |x(t_1)|$

$$= -\cancel{x(t_1)} - k_n v(t_1) + \cancel{x(t_1)}$$

$$\frac{\|x + k_n v\|_X - \|x\|_X}{k_n} \geq |t - t_2| > 0$$

$$\text{So } \liminf_{k_n \to 0^+} \frac{\|x + k_n v\|_X - \|x\|_X}{k_n} > 0 \quad \text{——} \quad (4)$$

Consider, $\|x + k_n v\|_X - \|x\|_X \geq |x(t_2) + k_n v(t_2)| - |x(t_2)|$

$$= |x(t_2)| - |x(t_2)| = 0$$

$$\text{So } \limsup_{k_n \to 0^-} \frac{\|x + k_n v\|_X - \|x\|_X}{k_n} \leq 0 \quad \text{——} \quad (5)$$

From (4) and (5), we get

$$\lim_{k_n \to 0} \frac{\|x + k_n v\|_X - \|x\|_X}{k_n} \text{ does not exist.}$$

So similarly we get a contradiction for existence of A_x on $C[0,1]$ as bounded linear functional and we are done!