Department of Mathematics

MPhil/PhD Coursework Examination

Topics in Analysis

(August 2023)

Time: 3 hr

Max. Marks : 70

Attempt FIVE questions in all. All symbols carry their usual meaning.

1. (a) State Browder's Theorem. Let

$$\mathcal{K} = \{x \in C[0,1] : 0 \le x(t) \le 1, x(0) = 0, x(1) = 1\}, F : x(t) \mapsto tx(t).$$

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Then show that (a) $F(\mathcal{K}) \subset \mathcal{K}$, (b) F is non-expansive and (c) $\mathcal{K}$ has no fixed point of F. Further, discuss the applicability of Browder's Theorem to this example. (7)

(b) Let $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous real function and assume there exists a constant $N > 0$ such that for any $t, \tau \in [a,b], z_1, z_2 \in \mathbb{R}, |K(t,\tau,z_1) - K(t,\tau,z_2)| \le N|z_1 - z_2|$. Let $h \in C[a,b]$ be fixed. Find conditions on $\lambda \in \mathbb{R}$ such that the integral equation $x(t) = \lambda \int_a^t K(t,\tau,x(\tau))d\tau + h(\tau)$ has a unique solution in $x \in C[a,b]$. (7)

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2. (a) Give examples of two algebras, one of which separates points and one which doesn't. Prove all your claims. (5)

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(b) State Stone-Weierstrass's Theorem for an algebra of complex continuous functions defined on a compact set K. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{A}$ be the set of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ where $N \in \mathbb{N}, c_n \in \mathbb{C}, \theta \in \mathbb{R}$. Determine which of the conditions of the above mentioned result are satisfied by $\mathcal{A}$. Further check whether or not the conclusion of the theorem holds? Give a reason in either case. (2+7)

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3. (a) Let $F : C[0,1] \to C[0,1]$ given by $(F(x))(t) = x(t) + \int_0^1 |x(st)|^2 ds$. Find the Frechet derivative of F at all those $x \in C[0,1]$ where it exists. Further, specify the domain and co-domain of $F'(x)$ for each x and also of F'. (10)

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(b) Let A be a bounded linear operator on a real Hilbert space H and define $F : H \to \mathbb{R}$ by $F(x) = \langle x, A^2 x\rangle$. Find the Frechet derivative of F at $x_0 \in H$. (4)

4. (a) Let f be defined on an open set Ω in the direct sum space $X = \sum_{i=1}^{n} \oplus X_i$ (where each X_j is a Banach space), and take values in a normed space Y, such that all partial derivatives $D_j f$ exist in Ω and are continuous at every point x_0 in Ω. Show that f is Frechet differentiable at x_0 and find an expression for the Frechet derivative of f at x_0. Illustrate this result with a non trivial example. (6+2)

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(b) State a general implicit function theorem for normed spaces. Illustrate it with a non-trivial example in which both the underlying spaces are not of the type $\mathbb{R}^n$ or $\mathbb{C}^n$. (6)

5. (a) Define $supp\,u$ for $u \in L^1_{loc}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open. Show by an example that the $supp\,u$ depends on the underlying set Ω. (3)

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(b) Show that for appropriate functions f, g

$$supp(f * g) \subset \overline{supp\, f + supp\, g}.$$

(5)

(c) Let $f \in L^1_{loc}(\Omega)$, with $\Omega \subset \mathbb{R}^n$. Show that if $\int_\Omega f\phi = 0$ for every $\phi \in C_c^\infty(\Omega)$, then $f = 0$. (6)

6. (a) Let $\Omega \subset \mathbb{R}^N$ be open. When is a function $f \in L^1_{loc}(\Omega)$ said to be weakly differentiable? Show that for $\Omega = (-1, 1)$ the functions (i) $f(t) = |t|$ and (ii) $f(t) = 2 + t + t^2$ are weakly differentiable and find their weak derivative. (2+3+3)

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(b) Show that if $f \in L^1_{loc}(\Omega)$ is weakly differentiable with weak derivative of f zero, then $f =$ constant. (6)

7. Let $\Omega = (a, b) \subset \mathbb{R}$. Define the spaces $W^{1,p}(a, b)$, $1 \leq < \infty$. Show that these spaces are separable Banach spaces with respect to appropriate norms. Further, give two non-trivial examples of elements in $W^{1,p}(a, b)$. (14)