Date
Page

Brouwer Fixed Point Theorem

Let K be non-empty, convex, closed and bounded subset of $\mathbb{R}^N$. Assume that $F: K \to K$ is continuous. Then F has a fixed point in K.

We have to do counter of Brouwer in infinite dimension space.

Example Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. (particularly ~~L^2 spa~~ $l^2(\mathbb{R})$ space).

Here for $x \in H$ $\langle x, e_n \rangle = x_n$.

Let $A: H \to H$ defined as

$$A\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n e_{n+1}$$

i.e. the right shift operator.

As it is easy to check that it is bounded linear operator.

$\therefore$ It is continuous.

Linearity is trivial

and $\|A\| = 1$. So bounded also.

Let K = closed unit ball.

~~Define~~ i.e. $K = \{x \in H \mid \|x\| \leq 1\}$

Then K is bounded, closed and convex. As closed balls are convex in normed space.

Define $F : K \to K$ ~~by~~ as
$$F(x) = (1 - \|x\|^2)^{1/2} e_1 + Ax$$

First to see wether F is well defined.

For $\|x\| \leq 1$
$$\|F(x)\| = \|(1-\|x\|^2)^{1/2} e_1 + Ax\|$$
(Here $\|x\| \leq 1$ So $(1-\|x\|^2)^{1/2}$ is w.d.)
$$= \|((1-\|x\|^2)^{1/2}, x_1, x_2, x_3, \ldots)\|$$
$$= \{(1-\|x\|^2)^{1/2}\}^2 + x_1^2 + x_2^2 + \ldots$$
$$= 1 - \|x\|^2 + \sum_{n=1}^{\infty} x_n^2$$
$$= 1 - \|x\|^2 + \|x\|^2 = 1$$
$$\Rightarrow F(x) \in K$$

Also F is continous as
$x \to (1-\|x\|^2)^{1/2} e_1$ is continous
[Since it is a composition of continous function $t \to \sqrt{t}$, $x \to 1 - \|x\|^2$.]

Let $x = \sum_{n=1}^{\infty} x_n e_n$ be a fixed point of F.

Then $Fx = x$
$$\Rightarrow ((1-\|x\|^2)^{1/2}, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$$
From here $x_n = x_{n+1}$ $\forall n \in \mathbb{N}$
and $x_1 = (1-\|x\|^2)^{1/2}$.

Date
Page

Now as $\|x\|^2 = \sum_{n=1}^{\infty} x_n^2 < \infty$

And as all x_n are same. Series becomes $\sum_{n=1}^{\infty} y^2$

This series is convergent iff $y = 0$
As y^2 is constant if it is non zero then series goes to $+\infty$.

$\Rightarrow$ $x_n = 0 \quad \forall n$. $\quad -(1)$

$\Rightarrow$ $x = 0$

$\Rightarrow$ $\|x\| = 0$

As $x_1 = (1 - \|x\|^2)^{1/2}$

$\Rightarrow$ $x_1 = 1$

Contradiction to (1).

Hence F does not have any fixed point.

3.1

Q.22 Let f be a differentiable map from one normed linear spaces into another. Let y be a point such that $f^{-1}(\{y\})$ contains no point x for which $f'(x)=0$. Prove that $f^{-1}(\{y\})$ contains no non void open set.

$f: X \to Y$. Claim:- $f^{-1}(\{y\})$ contains no nonvoid open set.

Soln Let $f^{-1}(\{y\})$ contains non void open set, means $f^{-1}(\{y\})$ contains non-empty open set (say U).

$f^{-1}(\{y\}) = \{x \in X \mid f(x)=y\}$.

Let $x_0 \in U$

Since U is open $\exists$ an open ball of radius $r>0$ centre at x_0 in U.

$B(x_0, r) \subseteq U$

As f is differentiable then $f'(x_0)$ exist call it A.

Then A is bounded linear operator.

s.t.

$$\lim_{h\to 0} \frac{\|f(x_0+h) - f(x_0) - A(h)\|_Y}{\|h\|_X} = 0$$

We can talk about limit in some nbhd. of 0.

So $$\lim_{\substack{h\to 0 \\ \|h\|<r}} \frac{\|f(x_0+h) - f(x_0) - A(h)\|_Y}{\|h\|_X} = 0$$

If $\|h\| < r$

then $x_0+h \in B(x_0, r)$

As $\|(x_0+h) - x_0\| = \|h\| < r$

Date
Page

Since $B(x_0, r) \subseteq U \subseteq f^{-1}(\{y\})$

$\Rightarrow f(x_0+h) = y \quad \forall \|h\| < r.$

Also $x_0 \in f^{-1}(\{y\})$

$$\Rightarrow \lim_{\substack{h \to 0 \\ \|h\| < r}} \frac{\| y - y - A(h) \|_Y}{\|h\|} = 0$$

$$\lim_{\substack{h \to 0 \\ \|h\| < r}} \frac{\| A(h) \|_Y}{\|h\|} = 0 \quad -(*)$$

Take $A = 0$ (zero operator)

then $(*)$ holds

By uniqueness of Frechet diff'ble

$f'(x_0) = 0$ which is a contradiction to the ~~claim~~ fact $f'(x) \neq 0$ $\forall x \in f^{-1}(\{y\})$.

Hence $f^{-1}(\{y\})$ contains no non void open set.

Our claim holds.

Q.23 If $f: \mathbb{R} \to \mathbb{R}^n$, what is the formula for $f'(x)$?

Soln: $f: \mathbb{R} \to \mathbb{R}^n$
then we can define it like
$f(x) = (f_1(x), f_2(x), \dots, f_n(x))$

Here we have to give formula of $f'(x)$ so we have to assume f is diff'ble.

Claim:- $A: \mathbb{R} \to \mathbb{R}^n$ is the ~~der~~ Frechet derivative of $f(x)$ and is defined as
$A(h) = (f_1'(x), \dots, f_n'(x))h$

Proof As f is diff'ble so all components of f i.e. f_i's are diff'ble.
and f_i's are from $\mathbb{R} \to \mathbb{R}$
and we know derivative in real line is defined as $f_i'(x)$.
and $\lim_{h \to 0} \frac{|f_i(x+h) - f_i(x) - f_i'(x).h|}{|h|} = 0.$

So here $\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A(h)\|}{|h|}$

~~~~
~~~~

Date
Page

$$\Rightarrow \lim_{h\to 0} \frac{\| (f_1(x+h), \cdots, f_n(x+h)) - (f_1(x), \cdots, f_n(x)) - (f_1'(x), \cdots f_n'(x)) \cdot h \|_{\mathbb{R}^n}}{|h|}$$

$$\Rightarrow \lim_{h\to 0} \frac{\| (f_1(x+h) - f_1(x) - f_1'(x)\cdot h, \cdots f_n(x+h) - f_n(x) - f_n'(x)\cdot h) \|_{\mathbb{R}^n}}{|h|}$$

$=$

In $\mathbb{R}^n$ there is euclidean norm.

$$\Rightarrow \lim_{h\to 0} \frac{1}{|h|} \left(\sum_{i=1}^{n} \left(f_i(x+h) - f_i(x) - f_i'(x)\cdot h \right)^2 \right)^{1/2}$$

$$\Rightarrow \left(\sum_{i=1}^{n} \left(\frac{f_i(x+h) - f_i(x) - f_i'(x)\cdot h}{|h|} \right)^2 \right)^{1/2} \quad - (*)$$

Here as $\forall i \in \{1, \cdots, n\}$

f_i is diff'ble

So $\frac{f_i(x+h) - f_i(x) - f_i'(x)\cdot h}{|h|} \to 0$

as $h \to 0$

And here $(*)$ is Sum of finite terms.

$\therefore (*) \to 0$ as $h \to 0$

Here A is also linear.

and $\| A(h) \| \le \left(\sum_{i=1}^{n} (f_i'(x))^2 \right)^{1/2} |h|$

So A is bdd also.

Hence our claim follows.