Thermal boundary layer –

Derivation of Energy Equation. page 265.

Consider, a motion of a compressible fluid of volume ΔV of density ρ. Let dQ amount of heat is added to the volume $\Delta V = dx\,dy\,dz$ of the fluid in time dt that is used in increasing total energy (internal energy + K.E if any) of the volume by amount dE_T and doing the work d. Therefore by first law of thermodynamics

$$\frac{dQ}{dt} = \frac{dE_T}{dt} + \frac{dW}{dt} \qquad (J/sec) \qquad -①$$

$$\frac{dE}{dt} = \frac{\partial E_T}{\partial t} + (\vec{w}\cdot\nabla)E_T,$$

Convection

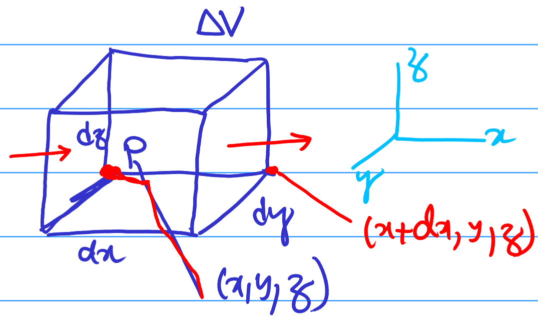
$\vec{w}$ is velocity of the fluid

$\vec{w} = u\hat{i} + v\hat{j} + w\hat{k}$ at (x,y,z)

We neglect radiation heat transfer and calculate the conduction heat transfer to the volume $\Delta V = dx\,dy\,dz$ by Fourier's law of heat conduction

heat flux ∝ temp. gradient

$$\frac{1}{A}\frac{dQ}{dt} = q = -K\frac{\partial T}{\partial n}$$



Amount of heat transferred to the volume ΔV through the face of area dydz ⊥r to x-axis per unit time

$$q(x) = -K\frac{\partial T}{\partial x}dy\,dz$$

Amount of heat leaving the volume ΔV through the parallel face of area dydz at $(x+dx, y, z)$

$$q(x+dx) = q(x) + \frac{\partial q}{\partial x}dx$$

$$= -K\frac{\partial T}{\partial x}dy\,dz + \frac{\partial}{\partial x}\left(-K\frac{\partial T}{\partial x}dy\,dz\right)dx$$

Amount of heat added to volume ΔV across faces (parallel) $\perp^r$ to x-axis per unit time

$$= q(x) - q(x+dx)$$
$$= \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)dx\,dy\,dz$$

Total amount of heat added by conduction into volume ΔV in time dt across all the six faces

$$dQ = dt.\Delta V\left[\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)\right] \quad -(2)$$

Total energy within the volume ΔV = K.E + E.E

$$= \frac{1}{2}\rho\Delta V(u^2+v^2+w^2) + e\rho\Delta V$$

where e is internal energy per unit mass

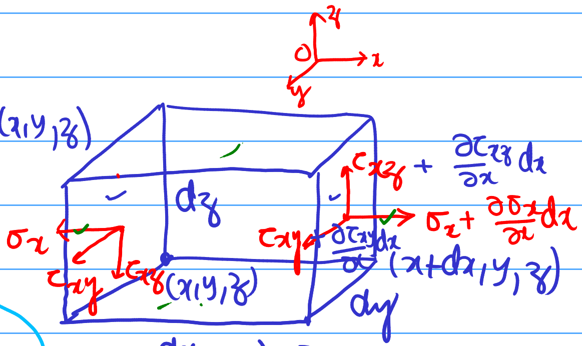
Rate of change of total energy within volume ΔV

$$\frac{dE_T}{dt} = \rho\Delta V\left[\frac{de}{dt}+\frac{1}{2}\frac{d}{dt}(u^2+v^2+w^2)\right] \quad -(3)$$

The work is done over the volume ΔV due to surface and body force. But we neglect the work done due to body force and only consider surface force

work/time $= \vec{F}.\vec{d}$/time $= Fd\cos\alpha$ $= Fd$, $\alpha = 0°$

$\alpha = 90°$, $W = 0$.



The work done by stress (normal, σ_x) on the face at (x,y,z) normal to x-axis per unit time

$= $ force $\times$ displacement

$= -\sigma_x.u\,dy\,dz$ ($-\sigma_x$ is along +ve x-axis)

rate of work done $= \vec{F}.$velocity

$$dW_x = -dy\,dz\left[-u\sigma_x + \left(u+\frac{\partial u}{\partial x}dx\right).\left(\sigma_x+\frac{\partial}{\partial x}(\sigma_x)dx\right)\right]$$

$O(4)$ and is negligible.

velocity component at face $(x+dx,y,z)$

$$= -\Delta V\frac{\partial}{\partial x}(u\sigma_x)$$

work is done on the volume

The total work done per unit time over volume ΔV from all sides by all surface forces

$$dW = -\Delta V\left[\frac{\partial}{\partial x}\left(u\sigma_x + v\tau_{xy} + w\tau_{xz}\right)\right.$$
$$+\frac{\partial}{\partial y}\left(u\tau_{yx} + v\sigma_y + w\tau_{yz}\right)$$
$$\left.+\frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\sigma_z\right)\right] \qquad \text{--- (4)}$$

Using eqn (2), (3) and (4) into eqn (1) we get the energy equation

$$\boxed{\rho\frac{de}{dt} + p\,\mathrm{div}\,\vec{w} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \mu\Phi}$$

where Φ represents the dissipation function given by

For two dim motion, $\bar{q} = u\hat{i} + v\hat{j}$, $u = u(x,y)$, $v = v(x,y)$

$$\Phi = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2$$
$$-\frac{2}{3}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2}_{\mathrm{div}\,\vec{w} = 0 \text{ for incompressible fluid}}$$

Simplifications

$pv = RT$, $v = \frac{1}{\rho}$

(1) For perfect gas $p = \rho RT$, $de = c_v\,dT$ and $dh = c_p\,dT$, where $h = e + pv$ = enthalpy, $c_p = c_v + R$

$p = \rho RT$

For incompressible fluid $de = c\,dT$ and $dh = de + \frac{dp}{\rho}$ ($\rho = \frac{1}{v}$, v = specific volume)

specific heat $c_v = \left(\frac{dQ}{dT}\right)_v$, $c_p = \left(\frac{dQ}{dT}\right)_p$

for such fluid $c_p = c_v = c$.

$$\frac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{w} + \vec{w}\cdot\nabla\rho = 0 \Rightarrow \rho\nabla\cdot\vec{w} + \frac{\partial\rho}{\partial t} + (\vec{w}\cdot\nabla)\rho = 0$$
$$\Rightarrow \nabla\cdot\vec{w} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

Also, we can write $\nabla\cdot\vec{w} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{d\rho}{dt}$ $\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{w}) = 0\right)$

Therefore, $c_p\frac{dT}{dt} = (c_v + R)\,dT = c_v\,dT + d(RT) = de + d\left(\frac{p}{\rho}\right) = \frac{de}{dt} + \frac{1}{\rho}\frac{dp}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}$

so. $\rho\frac{de}{dt} + p\,\mathrm{div}\,\vec{w} = \rho\left[c_p\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} + \frac{p}{\rho^2}\cancel{\frac{d\rho}{dt}}\right] - \frac{p}{\rho}\cancel{\frac{d\rho}{dt}} = \rho c_p\frac{dT}{dt} - \frac{dp}{dt}$

Finally, $$\boxed{\rho c_p\frac{dT}{dt} = \frac{dp}{dt} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \mu\Phi}$$

For constant thermal conductivity of $\underbrace{\text{incompressible fluid}}_{\mathrm{div}\,\vec{w}=0}$, $\boxed{\rho c\frac{dT}{dt} = k\nabla^2T + \mu\Phi}$