Lectures 7 & 8.

Lemma 1.19: Let U be a closed ball in a Banach space X. Let $F : U^{+} \to U$ be given, where U^{+} is an open set containing U. If

$$\sup\{\|F'(x)\| : x \in U\} < 1$$

then F has a unique fixed point in U.

Proof: Let $x, y \in U$.

Consider. $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$

$\subseteq U$.

F' exists on U; F is cts on U.

$\therefore$ By the mean value Th version 3,

$$\| F(x) - F(y) \| \leq \sup_{\xi \in [x,y]} \|F'(\xi)\| \, \|x-y\|$$

$$< \|x - y\|. \quad \text{——} \; (1)$$

Since (1) holds $\forall\, x, y \in U$, it follows that $F : U \longrightarrow U$ is a contraction. Since U is closed, it is itself a complete metric space

$\therefore$ By the Contraction Th. 1.18, F has a unique fixed point in U.

Classical Implicit Function Theorem. (1.20)

Let F be a C^1 function on the square

$$\{(x,y) : |x-x_0| \leq \delta, |y-y_0| \leq \delta\} \subseteq \mathbb{R}^2$$

[Note that C^1 means the derivative $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ exist and are continuous]

$F_1 := \frac{\partial F}{\partial x}$, $F_2 := \frac{\partial F}{\partial y}$

If

(i) $F(x_0, y_0) = 0$.

(ii) $F_2(x_0, y_0) \neq 0$.

Then $\exists$ a function f defined in a nbd N_1 of x_0 such that

(a) $y_0 = f(x_0)$

(b) f is continuously diff

(c) $F(x, f(x)) = 0$.
$\forall \ x \in \mathcal{N}$.

Furthermore,

$$f'(x) = -\frac{F_1(x,y)}{F_2(x,y)}.$$

when $y = f(x)$.

$$\|(x,y)\|_p := \left(\|x\|^p + \|y\|^p\right)^{1/p}..$$

$1 \leq p < \infty$.

$$\|(x,y)\|_\infty := \max\left(\|x\|, \|y\|\right)$$

__Definition 1.21__ Let X, Y, Z be Banach spaces. Let $F : X \times Y \longrightarrow Z$. be a map.
The Cartesian product $X \times Y$ is also a Banach space if we give it the norm

$$\|(x,y)\| = \|x\| + \|y\|.$$

$x \in X,\ y \in Y.$

If they exist, the partial derivatives of F

at $(x_0, y_0) \in X \times Y$.

are

bounded linear operators

$D_1 F(x_0, y_0)$ and $D_2 F(x_0, y_0)$

such that.

$$\lim_{h \to 0} \frac{\| F(x_0+h, y_0) - F(x_0, y_0) - D_1 F(x_0, y_0) h \|}{\|h\|} = 0 \qquad (h \in X)$$

and

$$\lim_{k \to 0} \frac{\| F(x_0, y_0+k) - F(x_0, y_0) - D_2 F(x_0, y_0) k \|}{\|k\|}. = 0 \qquad (k \in Y).$$

Thus $D_1F(x_0, y_0) \in \mathcal{L}(X, Z)$.

and $D_2F(x_0, y_0) \in \mathcal{L}(Y, Z)$.

$$F_i := D_i F$$

General Implicit Function

Theorem 1.22.

Let X, Y, Z be normed spaces, Y being assumed complete. Let Ω be an open set in $X \times Y$.

Let $F : \Omega \longrightarrow Z$

and $(x_0, y_0) \in \Omega$

Assume that

(i) F is continuous at (x_0, y_0)

(ii) $F(x_0, y_0) = 0$.

(iii) $D_2 F$ exists in Ω.

(iv) $D_2 F$ is cts at (x_0, y_0)

(v) $D_2 F(x_0, y_0)$ is invertible

Then there exists a function f defined on a nbd, say N, of x_0. such that

(a) $F(x, f(x)) = 0$, $x \in N$.

(b). $f(x_0) = y_0$

(c) f is continuous at x_0

(d) f is unique in the sense that any other function satisfying (a) $\longleftrightarrow$ (c) must agree with f on some nbd of x_0.

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Pf: We can assume that $(x_0, y_0) = (0,0)$.
Select $\delta > 0$ such that

$$\{(x,y) : \|x\| \leq \delta, \|y\| \leq \delta\} \subseteq \Omega.$$

$$\left\{(x, y) : \|x-0\| \leq \delta, \|y-0\| \leq \delta \right\} \subseteq \Omega.$$

$$\left[\begin{array}{l} (0,0) \subseteq \Omega \to \text{open in } X \times Y. \\ \therefore \exists\, \varepsilon > 0 \text{ st} \\ \quad B_\varepsilon(0,0) \subseteq \Omega \end{array}\right]$$

$$B_\varepsilon(0,0) = \{(x,y) : \|(x,y)-(0,0)\| < \varepsilon\}$$

$$= \{(x,y) : \|x\| + \|y\| < \varepsilon\}.$$

Now $\delta = \varepsilon/2$ works!!

Put $A := D_2F(0,0)$.

Then $A \in \mathcal{L}(Y,Z)$.

A is invertible by hyp. (iv) so that

$$A^{-1} \in \mathcal{L}(Z,Y).$$

For each x satisfying $\|x\| \leq \delta$, we define

$$G_x(y) := y - A^{-1} F(x,y), \quad y : \|y\| \leq \delta. \qquad \text{--- (1)}$$

Thus for a fixed x,

$$G_x : \{ y : \|y\| \leq \delta \} \longrightarrow Y.$$

We will show that for x in some subset of $\{x : \|x\| \leq \delta\}$, $\exists\, U$ . .

$$G_x : U \longrightarrow U, \quad \|G_x'(y)\| \leq 1/2 \quad \forall\, y \in U$$

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So that by Lemma 1
G_x has a unique fixed pt

We will prove that G_x has a fixed point.
Indeed if y^* is a fixed pt of G_x,
then $y^* = G_x(y^*) = y^* - A^{-1}F(x, y^*)$

$\Rightarrow \quad F(x, y^*) = 0$.

Now.

$$G_x'(y) = I - A^{-1} D_2 F(x, y).$$

L14.

$$= A^{-1}\left[D_2F(0,0) - D_2F(x,y)\right].$$

— (2)

By the continuity of D_2F at $(0,0)$, we can reduce δ, if necessary such that -

$$\left.\begin{array}{l}\{\|x\| \leq \delta \text{ and } \|y\| \leq \delta\} \\ \Rightarrow \|G_x'(y)\| \leq 1/2\end{array}\right\} \quad (3)$$

By cty of D_2F at $(0,0)$, $\exists$ $\delta' \leq \delta$, s.t.

$$\|D_2F(0,0) - D_2F(x,y)\| \leq \frac{1}{4\|A^{-1}\|}$$

where $\|x\| \le \delta'$, $\|y\| \le \delta'$.

Rename $\delta' \equiv \delta$.

Then by (2),
for $\|x\| \le \delta$, $\|y\| \le \delta$

$$\|G_x'(y)\| \le \frac{1}{4} < \frac{1}{2}.$$

Now
$$G_x(0) = -A^{-1} F(x,0).$$

$$= -A^{-1}\{F(x,0)$$

0.

Let $0 < \varepsilon < \delta$

By the continuity of F at $(0,0)$, we can find $\underline{\delta_\varepsilon} \in (0, \delta)$ so that

$$\|x\| \le \delta_\varepsilon \Rightarrow \|G_x(0)\| \le \varepsilon/2 \qquad \text{—} \; (4).$$

If $\|x\| \le \delta_\varepsilon$ and $\|y\| \le \varepsilon$.

then by the Mean Value Theorem $\overline{\underline{III}}$,

$0,0)$

$$\|G_n(y)\| \le \|G_n(0)\| + \|G_n(0) - G_n(y)\|$$

$$\le \varepsilon/2 + \sup_{0\le t\le 1} \|G_n'(ty)\| \cdot \|y\|.$$

(using 4)

$$\le \varepsilon/2 + \frac{1}{2}\cdot\|y\| \quad \text{(using 3)}$$

$$\le \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \text{--- } 5$$

Define
$$U = \{ y \in Y : \|y\| \leq \varepsilon \}$$
We have shown that for each x, with
$$\|x\| \leq \delta_\varepsilon .$$
the function G_x maps U into U.
(see 5).
Moreover, U is a closed subset of a complete

metric space.

$\therefore$ By Lemma 1.

G_x has a unique fixed point y in U.

We write

$$y := f(x)$$

Thus

$$f : \{x : \|x\| \le \delta_\varepsilon\} =: \mathcal{N} \longrightarrow U.$$

$$x \longrightarrow f(x).$$

$\therefore$ By our observation:

$$F(x, f(x)) = 0,$$

$\therefore$ Since $F(0,0) = 0$

$$G_0(0) = 0.$$

$\therefore$ By the uniqueness of fixed point

$$f(0) = 0.$$

<u>Continuity of f</u>: Since $\varepsilon > 0$

$0 < \varepsilon < \delta$ was arbitrary, we have the foll.

For each $\varepsilon > 0$, $\exists\ \delta_\varepsilon$ such that

$$\|x\| \leq \delta_\varepsilon \Rightarrow \|G_x(0)\| \leq \varepsilon/2$$

Then we showed that

$y = f(x) \in U$ ie

$$\|f(x)\| \leq \varepsilon\ ;$$

ie for $\varepsilon \in (0, \delta)$, $\exists\ \delta_\varepsilon > 0$

such that

$$\|x\| \leq \delta_\varepsilon \Rightarrow \|f(x)\| < \varepsilon.$$

$\Rightarrow f$ is cts at 0.

<u>Uniqueness</u>:

Suppose $\tilde{f}$ is another function defined on a nbd $\tilde{N}$ of 0. st

$\tilde{f}$ is cts at 0, $\tilde{f}(0) = 0$

$$F(x, \tilde{f}(x)) = 0, \quad x \in N.$$

If $0 < \varepsilon < \delta$, find $\theta > 0$ such that $\theta < \delta_\varepsilon$ and

$\|x\| \le \theta \Rightarrow \|\tilde{f}(x)\| < \varepsilon$

(by continuity of $\tilde{f}$).

$\therefore \tilde{f}(x) \in U$.

Thus $\tilde{f}(x)$, $f(x)$ are two fixed pts for G_x.

Since this is not possibl

$$\Rightarrow \quad f(x) = \tilde{f}(x)$$

when $\|x\| \leq \theta$.

$(x_0, y_0) = (0, 0)$.

Sugg.

$$G(x, y) := F(x_0 \overset{?}{+} x, \; y_0 \overset{?}{+} y).$$

Refine