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P2112

Q.23 (4.2 Page 185)

Let U be a closed set in a complete metric space X. Let f be a contraction map defined on U and taking values in X. Let α be a contraction constant, $x_0 \in U$, $f(x_0) = x_1$, $r = \frac{\alpha}{1-\alpha} d(x_0, x_1)$ and $B(x_1, r) \subseteq U$. Then Prove that f has a fixed point in $B(x_1, r)$.

$\Rightarrow$ If we consider $d(f(x), f(y)) \leq \alpha\, d(x, y)$, where $\alpha < 1$.

Then we have counter example.

- Let $X = \mathbb{R}$, $U = [0,1]$ and $f: U \to X$ is defined by

$$f(x) = \frac{x}{2} + \frac{1}{2}$$

Then
$$d(f(x), f(y)) = d\left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2} + \frac{1}{2}\right)$$
$$= \left| \frac{x}{2} + \frac{1}{2} - \left(\frac{y}{2} + \frac{1}{2}\right) \right|$$
$$= \frac{1}{2} |x - y|$$
$$= \frac{1}{2} d(x, y)$$

$$\Rightarrow d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$$

$\therefore$ f is contraction and $\alpha = \frac{1}{2}$.

Let $x_0 = 0 \in U$ then $f(0) = \frac{1}{2} = x_1$ and

$$r = \frac{1/2}{1 - \frac{1}{2}} d\left(x_0, \tfrac{1}{2}\right) = \frac{1}{2}$$

$\therefore$ $B(x_1, r) = B\left(\frac{1}{2}, \frac{1}{2}\right) = (0, 1)$ and f has NO fixed point in $(0, 1)$.

Let $\quad d(f(x), f(y)) < \lambda\, d(x, y) \quad \forall x, y \in X.$

Now,

$$d(f(x_1), x_1) = d(f(x_1), f(x_0))$$
$$< \lambda\, d(x_1, x_0)$$
$$= r(1-\lambda)$$

$\therefore \quad d(f(x_1), x_1) < r(1-\lambda)$

$\Rightarrow \exists\, r_0 < r$ such that $d(f(x_1), x_1) \leq r_0(1-\lambda)$ —— (*)

Now, $B[x_1, r_0] = \{x \in X : d(x_1, x) \leq r_0\}$

Let $x \in B[x_1, r_0]$

$\Rightarrow d(x_1, x) \leq r_0$ —— (**)

Claim: $f(x) \in B[x_1, r_0]$

$$d(x_1, f(x)) \leq d(f(x_1), f(x)) + d(f(x_1), x_1)$$
$$< \lambda\, d(x_1, x) + r_0(1-\lambda) \quad [\text{by } (*)]$$
$$\leq \lambda r_0 + r_0(1-\lambda) \quad [\text{by } (**)]$$
$$= r_0$$

$\therefore \quad d(x_1, f(x)) < r_0$

$\Rightarrow f(x) \in B[x_1, r_0]$

$\therefore \quad f : B[x_1, r_0] \to B[x_1, r_0]$, so by Banach Contraction theorem f has a fixed point in $B[x_1, r_0]$ and $B[x_1, r_0] \subseteq B(x_1, r)$.

Q.25 (4.2 ; P-186)

Let F be a contraction defined on a Banach space. Prove that $I-F$ is invertible and $(I-F)^{-1} = \lim_{n\to\infty} H_n$, where $H_0 = I$, $H_{n+1} = I + FH_n$

$\Rightarrow$

- $H_0(x) = I(x) = x$

- $H_1 = I + FH_0$

$\Rightarrow H_1(x) = (I + FH_0)(x)$

$= I(x) + FH_0(x)$

$= x + F(H_0(x))$

$= x + F(x)$

$= (I+F)(x)$

$\Rightarrow H_1 = I + F$

- $H_2 = I + FH_1$

$\Rightarrow H_2(x) = (I + FH_1)(x)$

$= I(x) + FH_1(x)$

$= x + F(H_1(x))$

$= x + F(x + F(x))$

$= x + F(x) + F^2(x)$ [Provided F is linear]

$= (I + F + F^2)(x)$

$\Rightarrow H_2 = I + F + F^2$

$\vdots$

$H_n = I + F + \cdots + F^n$

Now,

$$\|F(x) - F(y)\| \leq \lambda \|x-y\|$$

$$\Rightarrow \|F(x-y)\| \leq \lambda \|x-y\|$$

$\forall x, y \in \alpha$

$$\Rightarrow \|F(x)\| \leq \lambda \|x\|$$

$$\Rightarrow \|F\| \leq \lambda < 1$$

Let $n > m$

Now

$$\|H_n - H_m\| = \|F^{m+1} + F^{m+2} + \dots + F^n\|$$

$$\leq \|F^{m+1}\| + \|F^{m+2}\| + \dots + \|F^n\|$$

$$\leq \sum_{k=m+1}^{n} \|F^k\|$$

$$\leq \sum_{k=m+1}^{n} \|F\|^k$$

[$\because$ $L(X)$ is algebra, $\therefore$ $\|F^k\| \leq \|F\|^k$]

$$\leq \sum_{k=m+1}^{\infty} \|F\|^k$$

$$= \sum_{k=0}^{\infty} \|F\|^{k+m+1}$$

$$= \|F\|^{m+1} \sum_{k=0}^{\infty} \|F\|^{k}$$

$$= \|F\|^{m+1} \cdot \frac{1}{1-\|F\|} \quad [\because \|F\| < 1]$$

$$\therefore \quad \|H_n - H_m\| \leq \|F\|^{m+1} \cdot \frac{1}{1-\|F\|}$$

$\forall\, n > m$

$\Rightarrow \|H_n - H_m\| \longrightarrow 0$ as $m \to \infty$

$\therefore \{H_n\}$ is Cauchy seq. in $L(X)$.

Since $L(X)$ is a Banach space, $\{H_n\}$ is convergent in $L(X)$.

Let $\lim_{n\to\infty} H_n = H$

Now $(I-F)H_n = H_n - FH_n$

$$= \sum_{k=0}^{n} F^k - F\sum_{k=0}^{n} F^k$$

$$= \sum_{k=0}^{n} F^k - \sum_{k=0}^{n} F^{k+1}$$

$$= I - F^{n+1}$$

$$\Rightarrow \lim_{n\to\infty} (I-F)H_n = \lim_{n\to\infty} (I - F^{n+1})$$

$$\Rightarrow (I-F)H = I - \lim_{n\to\infty} F^{n+1}$$

$$= I - 0 = I$$

Explanation : $\lim_{n\to\infty} F^{n+1} = 0$

Since $\{H_n\}$ is cauchy, we have

$$\| H_n - H_{n+1} \| < \epsilon \qquad \forall n \geq N \text{ and } \epsilon > 0$$

$$\Rightarrow \| F^{n+1} \| < \epsilon \qquad \forall n \geq N \text{ and } \epsilon > 0$$

$$\Rightarrow \lim_{n\to\infty} F^{n+1} = 0$$

Similarly,

$$\begin{aligned} H_n (I - F) &= H_n - H_n F \\ &= \sum_{k=0}^{n} F^k - \left(\sum_{k=0}^{n} F^k\right) F \\ &= \sum_{k=0}^{n} F^k - \sum_{k=0}^{n} F^{k+1} \\ &= I - F^{n+1} \end{aligned}$$

$\therefore \quad H_n (I - F) = I - F^{n+1}$

$\Rightarrow \lim_{n \to \infty} H_n (I - F) = \lim_{n \to \infty} (I - F^{n+1})$

$\Rightarrow \quad H (I - F) = I - 0 \quad \left[\because \lim_{n \to \infty} F^{n+1} = 0 \right.$ explained ~~ea~~ $\left.\text{earlier}\right]$

$\therefore \quad H (I - F) = I = (I - F) H$

$\Rightarrow \quad (I - F)$ is invertable and

$$(I - F)^{-1} = H$$

i.e. $(I - F)^{-1} = \lim_{n \to \infty} H_n$