# R-07
# Convex and Nonsmooth Analysis

# Equivalence of two notions

Lemma $\mathrm{cl}f(x) \leq f(x), \forall x \in \mathbb{R}^n$.

Lemma Let $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function then the following are equivalent

(i) $\mathrm{cl}f(x) = \liminf_{y \to x} f(y), \forall x \in \mathbb{R}^n$,

(ii) $\mathrm{epi}(\mathrm{cl}f) = \mathrm{cl}(\mathrm{epi}f)$.

Proof (i) $\Rightarrow$ (ii)

Let $(x, r) \in \mathrm{epi}(\mathrm{cl}f)$. Then $\mathrm{cl}f(x) \leq r$. Hence

$$\liminf_{y \to x} f(y) \leq r$$

that is,

$$\sup_{\delta > 0} \inf_{y \in B_\delta(x)} f(y) \leq r.$$

For $\delta = \frac{1}{k}$, there exists $y_k \in B_\delta(x)$ such that $f(y_k) \leq r + \frac{1}{k}$. Thus $\left(y_k, r + \frac{1}{k}\right) \in \mathrm{epi}f$. Letting $k \to \infty$ we get $y_k \to x$. Hence, $(x, r) \in \mathrm{cl}(\mathrm{epi}f)$. Thus, $\mathrm{epi}(\mathrm{cl}f) \subseteq \mathrm{cl}(\mathrm{epi}f)$.

# continued

Let $(x,r) \in \mathrm{cl}(\mathrm{epi} f)$. Then there exists $(x_k, r_k) \in \mathrm{epi} f$ such that $(x_k, r_k) \to (x,r)$. As $f(x_k) \le r_k$ we have

$$\mathrm{cl} f(x) = \liminf_{y \to x} f(y) \le \liminf_{k \to \infty} f(x_k) \le \liminf_{k \to \infty} r_k = r.$$

Hence, $(x,r) \in \mathrm{epi}(\mathrm{cl} f)$, thus, $\mathrm{cl}(\mathrm{epi} f) \subseteq \mathrm{epi}(\mathrm{cl} f)$.

(ii) $\Rightarrow$ (i) Since epigraph of $\mathrm{cl} f$ is closed, it follows that $\mathrm{cl} f$ is lower semicontinuous. As $\mathrm{cl} f(x) \le f(x), \forall x \in \mathbb{R}^n$, we have

$$\mathrm{cl} f(x) \le \liminf_{y \to x} \mathrm{cl} f(y) \le \liminf_{y \to x} f(y). \qquad (1)$$

As $(x, \mathrm{cl} f(x)) \in \mathrm{epi}(\mathrm{cl} f) = \mathrm{cl}(\mathrm{epi} f)$, there exists $(x_k, r_k) \in \mathrm{epi} f$ such that $(x_k, r_k) \to \left(x, \mathrm{cl} f(x)\right)$. As $f(x_k) \le r_k$ we have

$$\liminf_{y \to x} f(y) \le \liminf_{k \to \infty} f(x_k) \le \liminf_{k \to \infty} r_k = \mathrm{cl} f(x). \qquad (2)$$

From (1) and (2) it follows that

$$\mathrm{cl} f(x) = \liminf_{y \to x} f(y), \forall x \in \mathbb{R}^n.$$