Chap 3, W. Cheney.

Lecture 1

Calculus on Banach Spaces. $\boxed{\text{Pg } 119 - 120}$

X, Y — normed spaces.

Definition 1.1 Let $D \subseteq X$ be an open set and

L2.

$f : D \xrightarrow{\subseteq X} Y$.

Let $x \in D$.

If $\exists$ a bounded linear map $A : X \to Y$ such that ($A \in \mathcal{B}(X,Y)$)

$$\boxed{\lim_{h \to 0} \frac{\| f(x+h) - f(x) - A(h) \|}{\|h\|} = 0}$$

($h \in X$, $x+h \in D$)

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then f is said to be Fréchet differentiable at x.

Furthermore, A is called the Fréchet derivative of f at x.

we write

$$f'(x) = A.$$

$$\boxed{\lim_{h\to 0}\left\| \frac{f(x+h)-f(x)-Ah}{h} \right\| = 0}$$

$\boxed{\text{A depends on } x}$

$$f : D(\subseteq X) \longrightarrow Y.$$

$x \in D$

$$f'(x) : X \longrightarrow Y$$

bdd linear

$$f' : D \longrightarrow \mathcal{B}(X, Y)$$

$$x \longrightarrow f'(x).$$

$\mathcal{B}(X, Y) \rightarrow$ Normed space of bdd linear operators from X to Y.

(assuming $f'(x)$ exists $\forall x \in D$).

Eg: Let $X = Y$.
and $T : X \to Y$ is a bdd linear operator.
Find the Frechet derivative of T at any $x \in X$.

Let $x \in X$.

$$f(x+h) - f(x) - A(h)$$

$$= T(x+h) - T(x) - Ah$$

$$= T(h) - Ah.$$

Choose $A = T$.!

Then

$$\frac{\| T(x+h) - T(x) - A(h) \|}{\| h \|}$$

$\longrightarrow 0$ as $h \to 0$.

This is true for every $x \in X$.

$\therefore T'(x) = T$

for each $x \in X$.

Theorem 1.2: Let $f : D \longrightarrow Y$, D is open subset of X, $x \in D$.

If f is differentiable at x, then the mapping A in Defn. 1.1 is uniquely defined. (it of course depends on f and x)

<u>Pf</u>: Suppose A_1, A_2 are two bdd linear maps from X to Y st

$$\frac{\|f(x+h) - f(x) - A_i(h)\|}{\|h\|}$$

$\longrightarrow 0$ as $h \to 0$

$i = 1, 2$.

$\therefore$ for every $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t.

$$\|f(x+h) - f(x) - A_i h\| < \varepsilon \|h\|$$

whenever $\|h\| < \delta$.

$i = 1, 2$.

$\therefore$

$$\|A_1 h - A_2 h\|$$

$$\leq \|A_1 h - (f(x+h) - f(x))\|$$

$$+ \|A_2 h - (f(x+h) - f(x))\|$$

$$< 2\varepsilon \|h\|.$$

whenever $\|h\| < \delta$.

$$\|A_1 h - A_2 h\| < 2\varepsilon \|h\|$$

when $\|h\| < \delta$ — (1)

??

$\therefore \| A_1 h - A_2 h \|$

$< 2\varepsilon \|h\|$

$\forall\, h \in X.$

For any, $h \in X$,

let $k = \dfrac{h\delta}{2\|h\|}$

$\in X$

and

$$\|k\| = \frac{\delta}{2} < \delta.$$

$\therefore$ by (1)

$$\|A_1 k - A_2 k\| \leq 2\varepsilon\|k\|$$

$$\therefore \|A_1 h - A_2 h\| \leq 2\varepsilon\|h\|.$$

$\forall h$.

$$\|A_1 - A_2\| < \varepsilon.$$

But $\varepsilon > 0$ was arbitrary.

$\therefore A_1 = A_2$.

<u>Th 1.3</u>: If f is bounded in a nbd. of $x \in D$. ($f : D \longrightarrow Y$)

open

and if a <u>linear map A</u>

satisfies

$$\frac{\|f(x+h)-f(x)-Ah\|}{\|h\|}$$

$\longrightarrow 0$ as $h \to 0$,

then A is bdd.

Hence A is the Fréchet derivative

of f at x.

Note: f bdd in a nbd of x, means, $\exists$ $r>0$, $K>0$ s.t.

$y \in B(x,r) \subseteq D$.

$\Rightarrow \|f(y)\| \leq K$.

Pf: Choose $\delta>0$ s.t. whenever $\|h\|<\delta$.

$$\|f(x+h)\| \le M \quad \text{— (1)}$$

and

$$\|f(x+h) - f(x) - Ah\| < \|h\| \ldots \quad \text{— (2)}$$

$\therefore$

$$\|Ah\| \le \|f(x+h) - f(x) - Ah\| + \|f(x+h) - f(x)\|$$

$$\le \|h\| + 2M.$$

whenever $\|h\| < \delta$.

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Using (1) & (2)

$\leq 2M + \delta$.

ie

$\|h\| < \delta$

$\Rightarrow \|Ah\| \leq 2M + \delta$.

$\therefore$ A is bounded.

Ex 1.4 :

Let $X = Y = \mathbb{R}$.
and $f(x) = \lambda x$.
usual derivative of f
at x_0 is λ.

Frechet derivative of
f at x_0 ?

a bdd linear op, $X \to Y$,

$$A(h) = \lambda \cdot h$$

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$$\frac{|f(x+h) - f(x) - Ah|}{|h|}$$

$$\left| \frac{f(x+h) - f(x) - A(h)}{h} \right|$$

$$\left| \frac{f(x+h) - f(x) - \lambda \cdot h}{h} \right|$$

$$= \left| \frac{f(x+h) - f(x)}{h} - \lambda \right| \longrightarrow 0$$