Ques (21) A map $f: X \to X$, where is a metric space, is said to be non expansive if $d(fx, fy) \leq d(x, y) \quad \forall x, y \in X$

P.t. if 'f' is nonexpansive and if the sequence $[f^n(x)]$ converges for each x, then the map $x \longmapsto \lim\limits_{n} f^n x$ is continuous.

Sol. $f: X \to X$ is non-expansive map

$$\Rightarrow d(fx, fy) \leq d(x, y) \quad \forall x, y \in X \quad \text{——— (1)}$$

Consider the map $T: X \to \bar{X}$ s.t $Tx = \lim\limits_{n \to \infty} f^n x$

(Above functn is well defined since $f^n x$ converges for each $x \in X$)

We have to show that the map Tx is continuous.

i.e, $d(Tx, Tx_0) < \varepsilon$ whenever $d(x, x_0) < \delta$ {for every $\varepsilon > 0$}

Using eq. (1) we can write –

$$d(f^n x, f^n x_0) \leq d(f^{n-1} x, f^{n-1} x_0) \leq \dots \leq d(f^2 x, f^2 x_0) \leq d(fx, fy) \leq d(x, y)$$

[implies that each f^n is continuous if we choose $\delta = \varepsilon \ \forall \varepsilon > 0$]

$\Rightarrow$ we can write for every $\varepsilon > 0 \ \exists \ \delta > 0$ s.t.

$$d(f^n x, f^n x_0) < \varepsilon/2 \quad \text{whenever} \quad d(x, x_0) < \delta$$

$$\Rightarrow \lim_{n \to \infty} d(f^n x, f^n x_0) \leq \varepsilon/2 \quad \text{whenever} \quad d(x, x_0) < \delta$$

$$\Rightarrow d\left(\lim_{n \to \infty} f^n x, \lim_{n \to \infty} f^n x_0\right) \leq \varepsilon/2 \quad \text{whenever} \quad d(x, x_0) < \delta$$

$$\Rightarrow d(Tx, Tx_0) \leq \varepsilon/2 \quad \text{whenever} \quad d(x, x_0) < \delta$$

$$\Rightarrow d(Tx, Tx_0) < \varepsilon \quad \text{whenever} \quad d(x, x_0) < \delta$$

Hence the function Tx is continuous.

Ques (22) Let C_0 be the Banach space of all real seq. $x = [x_1 x_2 \ldots]$ that tends to zero. The norm is defined by $\|x\| = \max_i |x_i|$.
Find all the fixed points of the mapping $f: C_0 \to C_0$ defined by

$$f(x) = [1, x_1, x_2, x_3, \ldots]$$

Show that f is non expansive and explain the significance of this example vis-a-vis the contract^n mapping Theorem.

Sol. for fixed points take, $f(x) = x$

$$f(x_1 x_2 x_3 \ldots) = (x_1 x_2 x_3 \ldots)$$
$$(1, x_1, x_2, x_3 \ldots) = (x_1 x_2 x_3 x_4 \ldots)$$

$\Rightarrow \quad x_1 = 1 = x_2 = x_3 = x_4 = \ldots$

$\Rightarrow \quad x = (1, 1, 1, 1, \ldots)$ be the fixed point of the given map.

But $(1, 1, 1, 1 \ldots) \not\to 0$ Hence $(1, 1, 1 \ldots) \notin C_0$

$\Rightarrow$ f has no fixed points in C_0.

To show that f is non-expansive -

$$|fx - fy| = |f(x_1 x_2 x_3 \ldots) - f(y_1 y_2 y_3 \ldots)|$$
$$= |(1, x_1, x_2 x_3 \ldots) - (1, y_1 y_2 \ldots)|$$
$$= |(0, (x_1 - y_1), (x_2 - y_2), (x_3 - y_3) \ldots)|$$

$\Rightarrow \quad \max |fx - fy| = \max_i |(x_i - y_i)| = \max_i |x - y|$

$\Rightarrow \quad \boxed{\|fx - fy\| = \|x - y\|}$

So that the given map is non-expansive.

Thus, a nonexpansive mapping in a complete metric space may not have of fixed point But for a contract^n mapping always has.