R-07
Convex and Nonsmooth Analysis

A Property of Closed convex Cone

Theorem A closed convex cone is the set of directions along which one can go upto infinity from any point of the cone.

Proof Let K be a closed convex cone and $x \in K$. We need to prove

$$K = \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\}.$$

For simplicity, let us represent the set on the right side by A.

Let $d \in K$. Since K is a convex cone we have

$$x + td \in K, \forall t > 0.$$

Conversely, let $d \in A$. Then $x + td \in K, \forall t > 0$. Hence,

$$d \in \frac{1}{t}(K - x), \forall t > 0.$$

As K is a cone we have

$$d \in K - \frac{1}{t}x, \forall t > 0.$$

As K is closed we have

$$d \in \mathrm{cl}K = K.$$

Cone

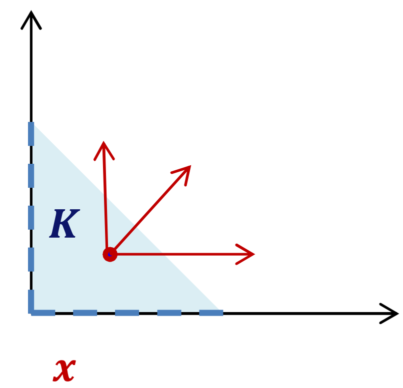
Theorem A closed convex cone is the set of directions along which one can go upto infinity from any point of the cone.

What if K is not closed?

$K \subseteq \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\}$.

Let $K = \text{int}\mathbb{R}^2_+$ then

$A = \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\} = \mathbb{R}^2_+$.

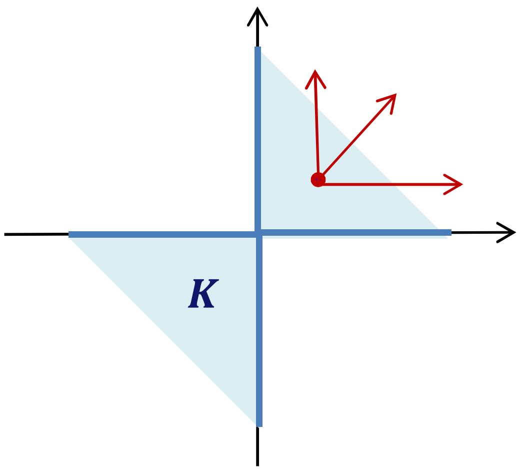


What if K is not convex?

$K \supseteq \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\}$.

Let $K = \mathbb{R}^2_+ \cup (-\mathbb{R}^2_+)$ then

for $x = (1,1) \in K$ we have $A = \mathbb{R}^2_+$.



Asymptotic Cone

Let C be a nonempty closed convex set in $\mathbb{R}^n$. For $x \in C$ let

$$C_\infty(x) := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}.$$

Theorem Let C be a nonempty closed convex set in $\mathbb{R}^n$ and $x \in C$. Then $C_\infty(x)$ is a closed convex cone.

Proof As $x \in C$ it is clear that $0 \in C_\infty(x)$.

Let $\{d_k\}$ be a sequence in $C_\infty(x)$ such that $d_k \to d$. As $d_k \in C_\infty(x)$ we have

$$x + td_k \in C, \forall t > 0.$$

Taking limit as $k \to \infty$ we have

$$x + td \in \mathrm{cl}C, \forall t > 0.$$

As C is a closed set we have $d \in C_\infty(x)$.

Claim $C_\infty(x)$ is a cone

Let $d \in C_\infty(x)$ and $\lambda > 0$. As $d \in C_\infty(x)$ we have

$$x + td \in C, \forall t > 0.$$

As $\lambda t > 0$ we have

$$x + t(\lambda d) \in C, \forall t > 0.$$

Hence, $\lambda d \in C_\infty(x)$.

continued

Claim $C_\infty(x)$ is convex

Let $d_1, d_2 \in C_\infty(x)$. Hence, $x + td_1 \in C, x + td_2 \in C, \forall t > 0.$

As C is convex for $\lambda \in [0,1]$ we have

$$\lambda(x + td_1) + (1 - \lambda)(x + td_2) \in C, \forall t > 0$$

which implies that

$$x + t(\lambda d_1 + (1 - \lambda)d_2) \in C, \forall t > 0.$$

Theorem Let C be a nonempty closed convex set in $\mathbb{R}^n$ and $x \in C$. The closed convex cone $C_\infty(x)$ does not depend on $x \in C$.

Proof Let $x_1, x_2 \in C, x_1 \neq x_2$.

Claim $C_\infty(x_1) \subseteq C_\infty(x_2)$

Let $d \in C_\infty(x_1)$ and $t > 0$. Let $\varepsilon \in (0,1)$. As $d \in C_\infty(x_1)$ we have

$$x_1 + \frac{t}{\varepsilon} d \in C.$$

As C is convex we have

$$\varepsilon\left(x_1 + \frac{t}{\varepsilon} d\right) + (1 - \varepsilon)x_2 \in C$$

that is,

$$\varepsilon x_1 + td + (1 - \varepsilon)x_2 \in C.$$

Taking limit as $\varepsilon \to 0+$, and using the fact that C is closed we have

$$x_2 + td \in \mathrm{cl}C = C$$

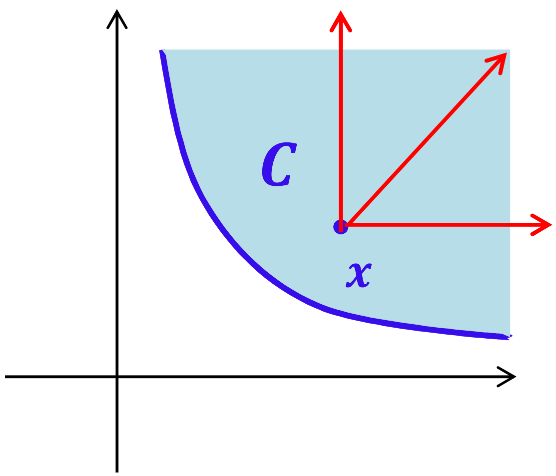
hence $d \in C_\infty(x_2)$.

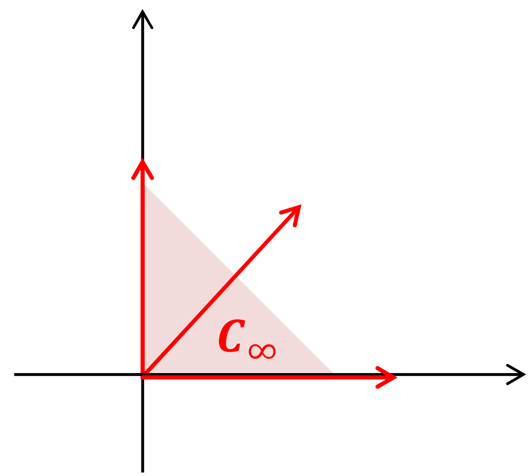
Asymptotic/Recession Cone

The asymptotic, or recession cone of a closed convex set C is the closed cone C_∞ defined as

$$C_\infty := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}$$

for any $x \in C$.

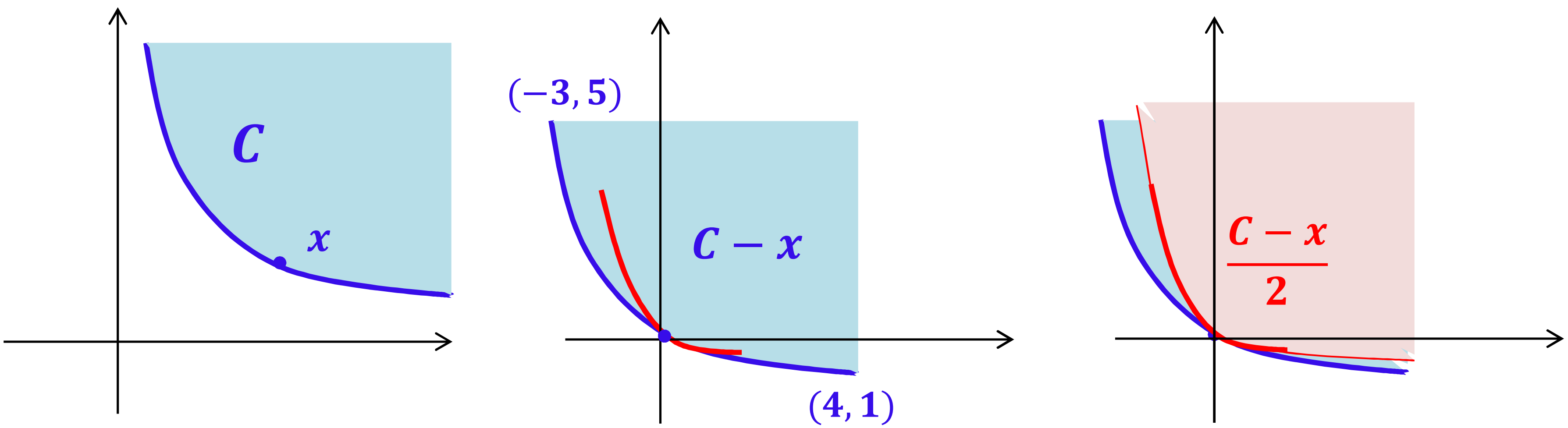




Asymptotic/Recession Cone

$$C_\infty = \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}$$

$$= \bigcap_{t>0} \frac{C-x}{t}.$$



Compactness of Asymptotic Cone

Theorem A closed convex set C is compact if and only if $C_\infty = \{0\}$.

Proof If C is bounded then $C_\infty = \{0\}$ as C cannot contain any nonzero direction.

Conversely, let $C_\infty = \{0\}$. Suppose on the contrary C_∞ is not bounded. Then there exists a sequence $\{x_k\} \subseteq C$ such that

$$\|x_k\| \to +\infty, x_k \neq 0.$$

Define $d_k = \frac{x_k}{\|x_k\|}$. As $\{d_k\}$ is bounded we can extract a convergent subsequence $\{d_{k_l}\}$ such that $d_{k_l} \to d$ with $\|d\| = 1$. Given $x \in C$ and $t > 0$ take k large such that $\|x_k\| \geq t$. Then

$$\left(1 - \frac{t}{\|x_{k_l}\|}\right)x + \frac{t}{\|x_{k_l}\|}x_{k_l} \in C.$$

Hence

$$x + td = \lim_{l\to\infty}\left[\left(1 - \frac{t}{\|x_{k_l}\|}\right)x + \frac{t}{\|x_{k_l}\|}x_{k_l}\right] \in \mathrm{cl}C = C.$$

Hence, $0 \neq d \in C_\infty$ which is a contradiction as $C_\infty = \{0\}$.

Nested Sequences

Theorem If $t_1 < t_2$ then $\frac{C-x}{t_1} \supseteq \frac{C-x}{t_2}$.

Proof Let $z \in \frac{C-x}{t_2}$. Then $z = \frac{y-x}{t_2}$ for some $y \in C$. Now

$$z = \frac{y-x}{t_2} = \frac{1}{t_1}\left(\frac{t_1(y-x)}{t_2} + x - x\right)$$

$$= \frac{1}{t_1}\left(\frac{t_1}{t_2}y + \frac{(t_2-t_1)}{t_2}x - x\right)$$

$$= \frac{1}{t_1}(y' - x)$$

where $y' = \frac{t_1}{t_2}y + \frac{(t_2-t_1)}{t_2}x \in C$ as C is a convex set. Hence,

$$z = \frac{1}{t_1}(y' - x) \in \frac{C-x}{t_1}.$$

Asymptotic Cone of Intersection of Convex Sets

Theorem If $\{C_j\}_{j\in J}$ is a family of closed convex sets such that $\bigcap_{j\in J} C_j \neq \emptyset$ then

$$\left(\bigcap_{j\in J} C_j\right)_\infty = \bigcap_{j\in J}\left(C_j\right)_\infty.$$

Proof Let $d \in \left(\bigcap_{j\in J} C_j\right)_\infty$. Then $x + td \in \bigcap_{j\in J} C_j$ for $t > 0$, $x \in \bigcap_{j\in J} C_j$. Hence,

$$x + td \in C_j \text{ for } t > 0, j \in J \text{ where } x \in C_j.$$

This implies that $d \in \left(C_j\right)_\infty$ for $j \in J$. Hence, $d \in \bigcap_{j\in J}\left(C_j\right)_\infty$.

Conversely, let $d \in \bigcap_{j\in J}\left(C_j\right)_\infty$. Let $x \in \bigcap_{j\in J} C_j$. Then $x \in C_j$ and $d \in \left(C_j\right)_\infty$ for $j \in J$. Hence, $x + td \in C_j$ for $t > 0, j \in J$. Hence,

$$x + td \in \bigcap_{j\in J} C_j \text{ for } t > 0$$

which implies that $d \in \left(\bigcap_{j\in J} C_j\right)_\infty$.

Asymptotic Cone and Affine Map

Theorem Let $A: \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. If C is a closed convex set in $\mathbb{R}^n$ and $A(C)$ is closed then

$$A(C_\infty) \subseteq [A(C)]_\infty.$$

Proof Let $d \in A(C_\infty)$. Then $d = A(p)$ where $p \in C_\infty$. Let $y \in A(C)$. Hence $y = A(x)$ for some $x \in C$. Now $x \in C, p \in C_\infty$ implies that

$$x + tp \in C \text{ for } t > 0.$$

This implies that

$$A(x + tp) \in A(C) \text{ for } t > 0.$$

As A is linear we have

$$A(x) + tA(p) \in A(C) \text{ for } t > 0.$$

Hence,

$$y + td \in A(C) \text{ for } t > 0$$

which implies that $d \in [A(C)]_\infty$.

What if A is not linear but affine? Give an example.

Give an example to show the containment is proper.

Asymptotic Cone and Affine Map

Theorem Let $A: \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. If D is a closed convex set in $\mathbb{R}^m$, with $A^{-1}(D) \neq \emptyset$ then

$$[A^{-1}(D)]_\infty = A^{-1}(D_\infty).$$

Proof Do it yourself.

Theorem If for $j = 1,2, \ldots m$, C_j are closed convex sets in $\mathbb{R}^{n_j}$ then

$$(C_1 \times C_2 \times \cdots \times C_m)_\infty = (C_1)_\infty \times (C_2)_\infty \times \ldots \times (C_m)_\infty.$$

Proof Do it yourself.

Avanindra Pratap Singh

1. Definition 2.3.1 of extreme points of a closed convex set with few examples;

2. Characterization in terms of convex combinations;

3. Proof of the fact that $C\setminus\{x\}$ is a convex set if x is an extreme point;

4. Example 2.3.2;

5. Prove that if C is a convex cone, then a nonzero $x \in C$ has no chance of being an extreme point;

6. Proposition 2.3.3;

7. Statement and illustration of Theorem 2.3.4.;

8. Example 2.3.5.

Note: Provide figures wherever possible for clarity

Archana Yadav

1. Definition 2.3.6 of face of a convex set;
2. Prove x is an extreme point of a closed convex set if and only if $\{x\}$ is a face of C;
3. Prove transmission of extremality;
4. Proposition 2.3.7;
5. Prove that if F' is a face of F, which is itself a face of C, then F' is a face of C;
6. Justify the remark that relative interiors of a convex set C for a partition of C;
7. Example of a set with no face of 1-dimension;
8. Definition 2.4.1 of supporting hyperplane;
9. Definition 2.4.2 of an exposed face of a convex set;
10. Proposition 2.4.3.

Note: Provide figures wherever possible for clarity

Veronica Khurana

1. State the properties of projection operator onto a subspace;

2. Discuss projection operator on a closed convex set and establish the existence and uniqueness of point of projection onto a closed convex set;

3. Theorem 3.1.1 with geometrical interpretation;

4. Justification for Remark 3.1.2;

5. Proposition 3.1.3;

6. Two consequences of Proposition 3.1.3.

Note: Provide figures wherever possible for clarity

Ashish Yadav

1. Definition 3.2.1 of polar of a cone;
2. Order reversing property of polarity;
3. Example 3.2.2(a), (b);
4. Polar of the cones $K = \{(x_1, x_2, z): z \geq \|x\|\}$ for l_1, l_2 and l_∞ norms with figures;
5. Proposition 3.2.3;
6. Properties before Theorem 3.2.5;
7. Theorem 3.2.5.

Note: Provide figures wherever possible for clarity

Bhawna

1. Theorem 4.1.1;
2. Corollary 4.1.3;
3. Support function;
4. Concepts of weak and proper separation;
5. Statement and illustration of Theorem 4.1.1.

Note: Provide figures wherever possible for clarity

Ruhi Sharma

1. Article 4.2(a)
2. Lemma 4.2.1;
3. Remark 4.2.2;
4. Proposition 4.2.3;

Note: Provide figures wherever possible for clarity

Rupleen Kaur Ahuja

1. Article 4.2(b);
2. Theorem 4.2.4;
3. Corollary 4.2.5;
4. Definition 4.2.6 of polyhedral sets;
5. Proposition 4.2.7;

Note: Provide figures wherever possible for clarity

Ajit Kumar

1. Article 4.2(b);
2. Theorem 4.2.4;
3. Corollary 4.2.5;
4. Definition 4.2.6 of polyhedral sets;
5. Proposition 4.2.7;

Note: Provide figures wherever possible for clarity

Anant Singh

1. Article 5.2 (After I introduce the concept in Article 5.1);
2. Proposition 5.2.1;
3. Definition 5.2.3 of normal cone;
4. Proposition 5.2.4;
5. Corollary 5.2.5;
6. Examples 5.2.6(a) and (b).

Note: Provide figures wherever possible for clarity

Jitendra Singh

1. Properties of tangent cone before Proposition 5.3.1;
2. Proposition 5.3.1 (i)-(iv);
3. Proposition 5.3.3.

Note: Provide figures wherever possible for clarity

Rohit Nageshwar

1. Definition 1.1.1;
2. Proposition 1.1.2;
3. Definitions 1.1.3-1.1.5;
4. Proposition 1.1.6;
5. Theorem 1.1.8;
6. Proposition 1.1.9.

Note: Provide figures wherever possible for clarity

Himanshu Bhatt

1. Proposition 1.2.1;
2. Notion of lower semicontinuity;
3. Proposition 1.2.2;
4. Definitions 1.2.3-1.2.4;
5. Proposition 1.2.5;
6. Proposition 1.2.6;
7. Notation 1.2.7.

Note: Provide figures wherever possible for clarity

Mohit

1. Article 1.3(a);
2. Article 1.3(b);
3. Article 1.3(c);
4. Article 1.3(d).

Note: Provide figures wherever possible for clarity

Gurudatt Rao

1. Article 1.3(g);
2. Theorem 1.3.1;
3. Proposition 2.1.1;
4. Proposition 2.1.2;
5. Proposition 2.1.5;
6. Proposition 2.1.6.

Note: Provide figures wherever possible for clarity

Majhar Alam

1. Definition 2.3.1;
2. Proposition 2.3.2;
3. Remark 2.3.3;
4. Properties before Remark 2.3.4;
5. Remark 2.3.4;
6. Example 2.3.5.

Note: Provide figures wherever possible for clarity

Monika

1. Lemma 3.1.1;
2. Theorem 3.1.2;
3. Remark 3.1.3.

Note: Provide figures wherever possible for clarity

Sachin Kumar

1. Theorem 3.1.5;
2. Theorem 4.1.1;
3. Definition 4.1.3;
4. Theorem 4.1.4.

Note: Provide figures wherever possible for clarity