## R-07 R-07<br>Convex and Nonsmooth Analysis

## A Property of Closed convex Cone

Theorem A closed convex cone is the set of directions along which one **A Property of Closed convex Con**<br>Theorem A closed convex cone is the set of directions a<br>can go upto infinity from any point of the cone.<br>Proof Let K be a closed convex cone and  $x \in K$ . We nee<br> $K = \{d \in \mathbb{R}^n : x + td \in K, \forall t$ 

**Proof** Let K be a closed convex cone and  $x \in K$ . We need to prove **Proof Let K** be a closed convex cone and  $x \in K$ . We ne<br>  $K = \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\}$ .<br>
For simplicity, let us represent the set on the right side<br>
Let  $d \in K$ . Since K is a convex cone we have<br>  $x + td \in K, \forall t > 0$ .<br>
Conv

$$
K = \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\}.
$$

For simplicity, let us represent the set on the right side by  $A$ .

Let  $d \in K$ . Since K is a convex cone we have<br> $x + td \in K$ ,  $\forall t > 0$ .

Conversely, let  $d \in A$ . Then  $x + td \in K$ ,  $\forall t > 0$ . Hence,

$$
d \in \frac{1}{t}(K - x), \forall t > 0.
$$

$$
d \in K - \frac{1}{t}x, \forall t > 0.
$$

As  $K$  is closed we have

$$
d \in \mathrm{cl} K = K.
$$

## Cone

**Cone**<br>
Theorem A closed convex cone is the set of directions along<br>
which one can go upto infinity from any point of the cone. **Cone**<br>Theorem A closed convex cone is the set of directions along<br>which one can go upto infinity from any point of the cone.<br>What if K is not closed?

What if  $K$  is not closed?  $K \subseteq \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\}.$ . Let  $K = \text{int} \mathbb{R}^2_+$  then  $A = \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\} = \mathbb{R}^2_+.$ .



## What if K is not convex?

 $K \supseteq \{d \in \mathbb{R}^n : x + td \in K, \forall t > 0\}.$ . Let  $K = \mathbb{R}_+^2 \cup (-\mathbb{R}_+^2)$  then  $\bar{+}$ ) then  $\binom{2}{+}$  then for  $x = (1,1) \in K$  we have  $A = \mathbb{R}^2_+$ .



## Asymptotic Cone

Let C be a nonempty closed convex set in  $\mathbb{R}^n$ . For  $x \in C$  let

 $\alpha(x) := \{u \in \mathbb{R}^n : x + \iota u \in C, \nu \}$  $n \cdot r + td \in C$   $\forall t > 0$ .

**Asymptotic Cone**<br>
Let *C* be a nonempty closed convex set in  $\mathbb{R}^n$ . For  $x \in C$  let<br>  $C_{\infty}(x) := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}.$ <br>
Theorem Let *C* be a nonempty closed convex set in  $\mathbb{R}^n$  and  $x \in C$ . Then<br>  $C_{\infty}(x)$ Theorem Let C be a nonempty closed convex set in  $\mathbb{R}^n$  and  $x \in C$ . Then **Asymptotic Cone**<br>
et *C* be a nonempty closed convex set in  $\mathbb{R}^n$ . For  $x \in C$  let<br>  $C_{\infty}(x) := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}.$ <br> *heorem* Let *C* be a nonempty closed convex set in  $\mathbb{R}^n$  and<br>  $C_{\infty}(x)$  is a closed

**Asymptotic Cone**<br>
Let *C* be a nonempty closed convex set in  $\mathbb{R}^n$ . For  $x \in C$  let<br>  $C_{\infty}(x) := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}$ .<br>
Theorem Let *C* be a nonempty closed convex set in  $\mathbb{R}^n$  and  $x \in C$ <br>  $C_{\infty}(x)$  is **Asymptotic Cone**<br>
Let *C* be a nonempty closed convex set in  $\mathbb{R}^n$ . For  $x \in C$  let<br>  $C_{\infty}(x) := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}$ .<br>
Theorem Let *C* be a nonempty closed convex set in  $\mathbb{R}^n$  and  $x \in C$ . Then<br>  $C_{\infty}(x$ have

 $x + td_k \in C$ ,  $\forall t > 0$ .

Taking limit as  $k \to \infty$  we have

 $x + td \in clC$ .  $\forall t > 0$ . As C is a closed set we have  $d \in C_{\infty}(x)$ . *Claim*  $C_{\infty}(x)$  is a cone Let  $d \in C_{\infty}(x)$  and  $\lambda > 0$ . As  $d \in C_{\infty}(x)$  we have  $x + td \in C$ ,  $\forall t > 0$ .

As  $\lambda t > 0$  we have

 $x + t(\lambda d) \in C$ ,  $\forall t > 0$ .

Hence,  $\lambda d \in C_{\infty}(x)$ .

## continued

*Claim*  $C_{\infty}(x)$  is convex

Let  $d_1, d_2 \in C_\infty(x)$ . Hence,  $x + td_1 \in C$ ,  $x + td_2 \in C$ ,  $\forall t > 0$ .<br>As C is convex for  $\lambda \in [0,1]$  we have

$$
\lambda(x + td_1) + (1 - \lambda)(x + td_2) \in C, \forall t > 0
$$

which implies that

$$
x + t(\lambda d_1 + (1 - \lambda)d_2) \in \mathcal{C}, \forall t > 0.
$$

Claim  $C_{\infty}(x)$  is convex<br>
Let  $d_1, d_2 \in C_{\infty}(x)$ . Hence,  $x + td_1 \in C$ ,  $x + td_2 \in C$ ,  $\forall t > 0$ .<br>
As C is convex for  $\lambda \in [0,1]$  we have<br>  $\lambda(x + td_1) + (1 - \lambda)(x + td_2) \in C$ ,  $\forall t > 0$ <br>
which implies that<br>  $x + t(\lambda d_1 + (1 - \lambda)d_2) \in C$ ,  $\forall$ 0.<br>  $> 0$ <br>
..<br>
and  $x \in C$ . The closed

**Continued**  
\n**Continued**  
\nLet 
$$
d_1, d_2 \in C_{\infty}(x)
$$
. Hence,  $x + td_1 \in C$ ,  $x + td_2 \in C$ ,  $\forall t > 0$ .  
\nAs C is convex for  $\lambda \in [0,1]$  we have  
\n
$$
\lambda(x + td_1) + (1 - \lambda)(x + td_2) \in C, \forall t > 0
$$
\nwhich implies that  
\n
$$
x + t(\lambda d_1 + (1 - \lambda)d_2) \in C, \forall t > 0.
$$
\n**Theorem** Let C be a nonempty closed convex set in  $\mathbb{R}^n$  and  $x \in C$ . The closed convex cone  $C_{\infty}(x)$  does not depend on  $x \in C$ .  
\n**Proof** Let  $x_1, x_2 \in C$ ,  $x_1 \neq x_2$ .  
\nClaim  $C_{\infty}(x_1) \subseteq C_{\infty}(x_2)$   
\nLet  $d \in C_{\infty}(x_1)$  and  $t > 0$ . Let  $\varepsilon \in (0,1)$ . As  $d \in C_{\infty}(x_1)$  we have  
\n
$$
x_1 + \frac{t}{\varepsilon}d \in C
$$
.

As  $C$  is convex we have

$$
\varepsilon \left( x_1 + \frac{t}{\varepsilon} d \right) + (1 - \varepsilon) x_2 \in C
$$

that is,

$$
\varepsilon x_1 + td + (1 - \varepsilon)x_2 \in \mathcal{C}.
$$

Taking limit as  $\varepsilon \to 0 +$ , and using the fact that C is closed we have  $x_2 + td \in clC = C$ 

hence  $d \in C_{\infty}(x_2)$ .

## Asymptotic/Recession Cone

**Asymptotic/Recession Cone**<br>The asymptotic, or recession cone of a closed convex set *C* is the<br>closed cone  $C_{\infty}$  defined as<br> $C_{\infty} := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}$ **Asymptotic/Recession Cone**<br>The asymptotic, or recession cone of a closed convex s<br>closed cone  $C_{\infty}$  defined as<br> $C_{\infty} := \{d \in \mathbb{R}^n : x + td \in C, \forall t > 0\}$ <br>for any  $x \in C$ .

for any  $x \in C$ .



## Asymptotic/Recession Cone

$$
C_{\infty} = \{ d \in \mathbb{R}^n : x + td \in C, \forall t > 0 \}
$$

$$
= \bigcap_{t > 0} \frac{c - x}{t}.
$$



## Compactness of Asymptotic Cone

Theorem A closed convex set C is compact if and only if  $C_{\infty} = \{0\}$ .

**Compactness of Asymptotic Cone**<br>Theorem A closed convex set C is compact if and only if  $C_{\infty} = \{0\}$ .<br>Proof If C is bounded then  $C_{\infty} = \{0\}$  as C cannot contain any nonzero<br>direction.<br>Conversely, let  $C_{\infty} = \{0\}$ . direction.

**Compactness of Asymptotic Cone**<br> *orem* A closed convex set *C* is compact if and only if  $C_{\infty} = \{0\}$ .<br> *f* If *C* is bounded then  $C_{\infty} = \{0\}$  as *C* cannot contain any nonzero<br>
conversely, let  $C_{\infty} = \{0\}$ . Supp **Compactness of Asymptotic Cone**<br>
Theorem A closed convex set C is compact if and only if  $C_{\infty} = \{0\}$ .<br>
Proof If C is bounded then  $C_{\infty} = \{0\}$  as C cannot contain any nonzero<br>
direction.<br>
Conversely, let  $C_{\infty} = \{0$ 

$$
||x_k|| \to +\infty, x_k \neq 0.
$$

Define  $d_k = \frac{\lambda_k}{\|\mathbf{x}_k\|}$ . As  $\{d_k\}$  i  $\frac{x_k}{x_k}$  As  $\{d\}$  is boung  $\|x_k\|$ **IMEXALLE SOMET ASSUMPT CORE**<br>
convex set C is compact if and only if  $C_{\infty} = \{0\}$ .<br>
Inded then  $C_{\infty} = \{0\}$  as C cannot contain any nonzero<br>
ort  $C_{\infty} = \{0\}$ . Suppose on the contrary  $C_{\infty}$  is not bounded.<br>
a seq **Compactness or Asymptotic Cone**<br>
Theorem A closed convex set C is compact if and only if  $C_{\infty} = \{0\}$ .<br>
Proof If C is bounded then  $C_{\infty} = \{0\}$  as C cannot contain any nonzero<br>
direction.<br>
Conversely, let  $C_{\infty} = \{0$ *em* A closed convex set *C* is compact if and only if  $C_{\infty}$  = <br>
If *C* is bounded then  $C_{\infty} = \{0\}$  as *C* cannot contain<br>
on.<br>
bonversely, let  $C_{\infty} = \{0\}$ . Suppose on the contrary  $C_{\infty}$  is is<br>
there exists a

$$
\left(1 - \frac{t}{\|x_{k_l}\|}\right) x + \frac{t}{\|x_{k_l}\|} x_{k_l} \in C.
$$

**Hence** 

$$
x + td = \lim_{l \to \infty} \left[ \left( 1 - \frac{t}{\|x_{k_l}\|} \right) x + \frac{t}{\|x_{k_l}\|} x_{k_l} \right] \in clC = C.
$$

Hence,  $0 \neq d \in C_{\infty}$  which is a contradiction as  $C_{\infty} = \{0\}$ .

## Nested Sequences

**Nested Sequences**

\n**Theorem** If 
$$
t_1 < t_2
$$
 then  $\frac{c-x}{t_1} \supseteq \frac{c-x}{t_2}$ .

\n**Proof** Let  $z \in \frac{c-x}{t_2}$ . Then  $z = \frac{y-x}{t_2}$  for some  $y \in C$ . Now

\n
$$
z = \frac{y-x}{t_2} = \frac{1}{t_1} \left( \frac{t_1(y-x)}{t_2} + x - x \right)
$$
\n
$$
= \frac{1}{t_1} \left( \frac{t_1}{t_2} y + \frac{(t_2 - t_1)}{t_2} x - x \right)
$$
\n
$$
= \frac{1}{t_1} (y' - x)
$$

where  $y' = \frac{t_1}{t}y + \frac{(t_2 - t_1)}{t}x \in C$  a 2  $\overline{t}_2$  $2^{-l_1}$   $\vee$   $\subset$   $\cap$   $\cap$   $\subset$   $\cap$   $\cap$ మ as  $C$  is a convex set. Hence,  $\iota_1$   $\iota_2$ .

## Asymptotic Cone of Intersection of Convex Sets

Theorem If  ${C_j}_{i \in I}$  is a family of closed convex sets such that  $\bigcap_{i\in I}C_i\neq\emptyset$  then **iptotic Cone of Intersection of**<br>  $\{C_j\}_{j\in J}$  is a family of closed convex<br>
i then<br>  $(\bigcap_{j\in J} C_j)_{\infty} = \bigcap_{j\in J} (C_j)_{\infty}$ .<br>  $d \in (\bigcap_{j\in J} C_j)_{\infty}$ . Then  $x + td \in$ <br>  $x + td \in C_j$  for  $t > 0, j \in J$  wher<br>
that  $d \in (C_j)$ , for  $i \$ 

$$
\left(\bigcap_{j\in J}C_j\right)_{\infty}=\bigcap_{j\in J}\left(C_j\right)_{\infty}.
$$

**Asymptotic Cone of Intersection of Convex Sets**<br>
Theorem If  $\{C_j\}_{j \in J}$  is a family of closed convex sets such that<br>  $\bigcap_{j \in J} C_j \neq \emptyset$  then<br>  $\bigcap_{j \in J} C_j \big)_{\infty} = \bigcap_{j \in J} (C_j)_{\infty}$ .<br>
Proof Let  $d \in \bigcap_{j \in J} C_j \big)_{\infty$ 

 $x + td \in C_i$  for  $t > 0, j \in J$  where  $x \in C_i$ .

 $\left(\bigcap_{j\in J} C_j\right)_{\infty} = \bigcap_{j\in J} \left(\frac{C_j}{C_j}\right)_{\infty}.$ <br> *Proof* Let  $d \in \left(\bigcap_{j\in J} C_j\right)_{\infty}$ . Then  $x + td \in \bigcap_{j\in J} C_j$  for  $t > 0$ ,  $x \in \bigcap_{j\in J} C_j$ . Hence,  $x + td \in C_j$  for  $t > 0, j \in J$  where  $x \in C_j$ .<br>
This implies that  $d \in \left(\$  $d \in (C_i)$  for  $j \in J$ . Hence,  $x + td \in C_j$  for  $t > 0, j \in J$ . Hence,  $x + td \in \bigcap_{i \in I} C_i$  for  $t > 0$ which implies that  $d \in (\bigcap_{i \in I} C_i)_{\sim}.$ 

## Asymptotic Cone and Affine Map

Theorem Let  $A: \mathbb{R}^n \to \mathbb{R}^m$  be a linear operator. If C is a closed convex set in  $\mathbb{R}^n$  and  $A(C)$  is closed then

$$
A(C_{\infty}) \subseteq [A(C)]_{\infty}.
$$

**Asymptotic Cone and Affine Map**<br>
Theorem Let  $A: \mathbb{R}^n \to \mathbb{R}^m$  be a linear operator. If *C* is a closed convex<br>
set in  $\mathbb{R}^n$  and  $A(C)$  is closed then<br>  $A(C_{\infty}) \subseteq [A(C)]_{\infty}$ .<br>
Proof Let  $d \in A(C_{\infty})$ . Then  $d = A(p)$ **Asymptotic Cone and Affine Map**<br>
Theorem Let  $A: \mathbb{R}^n \to \mathbb{R}^m$  be a linear operator. If  $C$  is a closed convex<br>
set in  $\mathbb{R}^n$  and  $A(C)$  is closed then<br>  $A(C_\infty) \subseteq [A(C)]_\infty$ .<br>
Proof Let  $d \in A(C_\infty)$ . Then  $d = A(p)$  whe  $x + tp \in C$  for  $t > 0$ .

This implies that

$$
A(x + tp) \in A(C) \text{ for } t > 0.
$$

As  $\overline{A}$  is linear we have

$$
A(x) + tA(p) \in A(C) \text{ for } t > 0.
$$

Hence,

$$
y + td \in A(C) \text{ for } t > 0
$$

which implies that  $d \in [A(C)]_{\infty}$ .

What if  $A$  is not linear but affine? Give an example.

Give an example to show the containment is proper.

## Asymptotic Cone and Affine Map

Theorem Let  $A: \mathbb{R}^n \to \mathbb{R}^m$  be a linear operator. If D is a closed convex set in  $\mathbb{R}^m$ , with  $A^{-1}(D) \neq \emptyset$  then **Asymptotic Cone and Aff**<br>
Theorem Let  $A: \mathbb{R}^n \to \mathbb{R}^m$  be a linear operator<br>
set in  $\mathbb{R}^m$ , with  $A^{-1}(D) \neq \emptyset$  then<br>  $[A^{-1}(D)]_{\infty} = A^{-1}(D_{\infty})$ <br>
Proof Do it yourself.<br>
Theorem If for  $j = 1, 2, ..., m$ ,  $C_j$  are close **Asymptotic Cone and Affine Map**<br>
Theorem Let  $A: \mathbb{R}^n \to \mathbb{R}^m$  be a linear operator. If  $D$  is a closed convex<br>
set in  $\mathbb{R}^m$ , with  $A^{-1}(D) \neq \emptyset$  then<br>  $[A^{-1}(D)]_{\infty} = A^{-1}(D_{\infty})$ .<br>
Proof Do it yourself.<br>
Theorem

$$
[A^{-1}(D)]_{\infty} = A^{-1}(D_{\infty}).
$$

Theorem If for  $j = 1, 2, ..., m$ ,  $C_j$  are closed convex sets in  $\mathbb{R}^{n_j}$  then

$$
(C_1 \times C_2 \times \cdots \times C_m)_{\infty} = (C_1)_{\infty} \times (C_2)_{\infty} \times \cdots \times (C_m)_{\infty}.
$$

Proof Do it yourself.

**Avanindra Pratap Singh<br>extreme points of a closed convex set with few** 1. Definition 2.3.1 of extreme points of a closed convex set with few examples; **2. Avanindra Pratap Singh**<br>2. Definition 2.3.1 of extreme points of a closed convex set with few<br>2. Characterization in terms of convex combinations;<br>3. Proof of the fact that  $C \setminus \{x\}$  is a convex set if x is an extre **3. Proof of the fact that is a convex set with few**<br>3. Definition 2.3.1 of extreme points of a closed convex set with few<br>3. Characterization in terms of convex combinations;<br>3. Proof of the fact that  $C \setminus \{x\}$  is a co **Avanindra Pratap Si**<br>1. Definition 2.3.1 of extreme points of a close<br>examples;<br>2. Characterization in terms of convex combir<br>3. Proof of the fact that  $C \setminus \{x\}$  is a convex set<br>4. Example 2.3.2;<br>5. Prove that if *C* i **4.** Definition 2.3.1 of extreme points of a closed convex set with few examples;<br>2. Characterization in terms of convex combinations;<br>3. Proof of the fact that  $C \setminus \{x\}$  is a convex set if x is an extreme point;<br>4. Exa **Avanindra Pratap Singh**<br>
1. Definition 2.3.1 of extreme points of a closed convex set wit<br>
examples;<br>
2. Characterization in terms of convex combinations;<br>
3. Proof of the fact that  $C \{x\}$  is a convex set if x is an ex 1. Definition 2.3.1 of extreme points of a closed examples;<br>2. Characterization in terms of convex combinati<br>3. Proof of the fact that  $C \setminus \{x\}$  is a convex set if x<br>4. Example 2.3.2;<br>5. Prove that if *C* is a convex co

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examples;<br>2. Characterization in terms of convex combinations;<br>3. Proof of the fact that  $C \setminus \{x\}$  is a convex set if x is an extreme point;<br>4. Example 2.3.2;<br>5. Prove that if C is a convex cone, then a nonzero  $x \in C$  h 2. Characterization in terms of convex combir<br>
3. Proof of the fact that  $C \setminus \{x\}$  is a convex set<br>
4. Example 2.3.2;<br>
5. Prove that if  $C$  is a convex cone, then a<br>
chance of being an extreme point;<br>
6. Proposition 2.3 4. Example 2.3.2;<br>5. Prove that if *C* is a convex cone, then a nonzero  $x \in C$  has no<br>chance of being an extreme point;<br>6. Proposition 2.3.3;<br>7. Statement and illustration of Theorem 2.3.4.;<br>8. Example 2.3.5.<br>Note: Provid

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1. Definition 2.3.6 of face of a convex set;

**Archana Yadav**<br>of a convex set;<br>pint of a closed convex set if and only if {x} **Archana Yadav**<br>2. Definition 2.3.6 of face of a convex set;<br>2. Prove x is an extreme point of a closed convex set if and only if  $\{x\}$ <br>is a face of C;<br>3. Prove transmission of extremality; **Archana Yada**<br> **1.** Definition 2.3.6 of face of a convex set;<br>
2. Prove x is an extreme point of a closed<br>
is a face of C;<br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7; **Archana Yadav**<br> **1.** Definition 2.3.6 of face of a convex set;<br>
2. Prove x is an extreme point of a closed convex set if and<br>
is a face of C;<br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7;<br>
5. Prove that i **Archana Yadav**<br>
1. Definition 2.3.6 of face of a convex set;<br>
2. Prove x is an extreme point of a closed conve:<br>
is a face of C;<br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7;<br>
5. Prove that if  $F'$  is a f **Archana Y**<br> **1.** Definition 2.3.6 of face of a convex s<br>
2. Prove x is an extreme point of a clos<br>
is a face of C;<br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7;<br>
5. Prove that if  $F'$  is a face of  $F$ , wh 1. Definition 2.3.6 of face of a convex set;<br>2. Prove x is an extreme point of a closed c<br>is a face of  $C$ ;<br>3. Prove transmission of extremality;<br>4. Proposition 2.3.7;<br>5. Prove that if  $F'$  is a face of  $F$ , which is its<br>

**1.** Definition 2.3.6 of face of a convex set;<br>
2. Prove x is an extreme point of a closed convex set if and only if  $\{x\}$ <br>
is a face of C;<br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7;<br>
5. Prove that if **EXECT 1.** Definition 2.3.6 of face of a convex set;<br>
2. Prove x is an extreme point of a closed convex set if and only if  $\{x\}$ <br>
is a face of C;<br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7;<br>
5. Prove t 2. Prove x is an extreme point of a closed convex set if and only if  $\{x \in \mathbb{R}^n : x \in \mathbb{R}^n : x \in \mathbb{R}^n : x \in \mathbb{R}^n \}$ <br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7;<br>
5. Prove that if  $F'$  is a face of Is a face of *C*;<br>
3. Prove transmission of extremality;<br>
4. Proposition 2.3.7;<br>
5. Prove that if  $F'$  is a face of  $F$ , which is itself a face of  $C$ , then  $F'$ <br>
face of  $C$ ;<br>
6. Justify the remark that relative interior 3. Prove transmission of extremality;<br>4. Proposition 2.3.7;<br>5. Prove that if  $F'$  is a face of  $F$ , which is itself a face of  $C$ , then  $F'$  is a<br>face of  $C$ ;<br>6. Justify the remark that relative interiors of a convex set 4. Proposition 2.3.7;<br>
5. Prove that if  $F'$  is a face of  $F$ , which is itself a fa<br>
face of  $C$ ;<br>
6. Justify the remark that relative interiors of a<br>
partition of  $C$ ;<br>
7. Example of a set with no face of 1-dimension;<br>
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From the remark that relative interiors of a convex set  $C$  for a partition of  $C$ ;<br>
7. Example of a set with no face of 1-dimension;<br>
8. Definition 2.4.1 of supporting hyperplane;<br>
9. Definition 2.4.2 of an exposed face

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## Veronica Khurana

1. State the properties of projection operator onto a subspace;

**Veronica Khurana**<br>2. Discuss projection operator operator onto a subspace;<br>2. Discuss projection operator on a closed convex set and establish<br>the existence and uniqueness of point of projection onto a closed<br>convex set; **Veronica Khurana**<br>1. State the properties of projection operator onto a subspace;<br>2. Discuss projection operator on a closed convex set and establish<br>the existence and uniqueness of point of projection onto a closed<br>conve **Veronica Khur<br>
1.** State the properties of projection oper<br>
2. Discuss projection operator on a close<br>
the existence and uniqueness of point<br>
convex set;<br>
3. Theorem 3.1.1 with geometrical interp<br>
4. Justification for Rem **Solution:**<br> **State the properties of projection operator onto a subspace;<br>
2. Discuss projection operator on a closed convex set and establish<br>
the existence and uniqueness of point of projection onto a closed<br>
convex set Veronica Khurana**<br>
1. State the properties of projection operator onto a subspace<br>
2. Discuss projection operator on a closed convex set and<br>
the existence and uniqueness of point of projection onto<br>
convex set;<br>
3. Theor **Veronica Khurana**<br>1. State the properties of projection operator on<br>2. Discuss projection operator on a closed con-<br>the existence and uniqueness of point of proj<br>convex set;<br>3. Theorem 3.1.1 with geometrical interpretatio 1. State the properties of projection operator onto a subspace;<br>2. Discuss projection operator on a closed convex set and establish<br>the existence and uniqueness of point of projection onto a closed<br>convex set;<br>3. Theorem 3 the existence and uniqueness of point of projection onto a closed<br>convex set;<br>3. Theorem 3.1.1 with geometrical interpretation;<br>4. Justification for Remark 3.1.2;<br>5. Proposition 3.1.3;<br>6. Two consequences of Proposition 3.

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# **Ashish Yadav**<br>of a cone;<br>of polarity:

- 1. Definition 3.2.1 of polar of a cone;
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**Ashish Yadav**<br> **2.** Definition 3.2.1 of polar of a cone;<br>
2. Order reversing property of polarity;<br>
3. Example 3.2.2(a), (b);<br>
4. Polar of the cones  $K = \{(x_1, x_2, z): z \ge ||x||\}$  for  $l_1, l_2$  and **Ashish Yadav**<br> **1.** Definition 3.2.1 of polar of a cone;<br>
2. Order reversing property of polarity;<br>
3. Example 3.2.2(a), (b);<br>
4. Polar of the cones  $K = \{(x_1, x_2, z): z \ge ||x||\}$  fo<br>
norms with figures; **4.** Definition 3.2.1 of polar of a cone;<br>
2. Order reversing property of polarity;<br>
3. Example 3.2.2(a), (b);<br>
4. Polar of the cones  $K = \{(x_1, x_2, z): z \ge ||x||\}$  for  $l_1, l_2$  and  $l_\infty$ <br>
norms with figures;<br>
5. Proposition **Ashish Yadav**<br>
1. Definition 3.2.1 of polar of a cone;<br>
2. Order reversing property of polarity;<br>
3. Example 3.2.2(a), (b);<br>
4. Polar of the cones  $K = \{(x_1, x_2, z) : z \ge$ <br>
norms with figures;<br>
5. Proposition 3.2.3;<br>
6. Prop **Ashish Yadav**<br>1. Definition 3.2.1 of polar of a cone;<br>2. Order reversing property of polarity;<br>3. Example 3.2.2(a), (b);<br>4. Polar of the cones  $K = \{(x_1, x_2, z): z \ge ||x$ <br>norms with figures;<br>5. Proposition 3.2.3;<br>6. Properties **ASNISN YAGAV**<br> **1.** Definition 3.2.1 of polar of a cone;<br> **2.** Order reversing property of polarity;<br> **3.** Example 3.2.2(a), (b);<br> **4.** Polar of the cones  $K = \{(x_1, x_2, z) : z \ge ||x||\}$  for  $l_1, l_2$  ar<br>
norms with figures;<br> 1. Definition 3.2.1 of polar of a cone;<br>2. Order reversing property of polarity;<br>3. Example 3.2.2(a), (b);<br>4. Polar of the cones  $K = \{(x_1, x_2, z) : z \ge$ <br>norms with figures;<br>5. Proposition 3.2.3;<br>6. Properties before Theorem 3. Example 3.2.2(a), (b);<br>4. Polar of the cones  $K = \{(x_1, x_2, z) : z \ge ||x||\}$  for  $l_1, l_2$  and  $l_\infty$ <br>norms with figures;<br>5. Proposition 3.2.3;<br>6. Properties before Theorem 3.2.5;<br>7. Theorem 3.2.5.<br>Note: Provide figures wher

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## Bhawna

- 1. Theorem 4.1.1;
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- **Bhawna**<br>1. Theorem 4.1.1;<br>2. Corollary 4.1.3;<br>3. Support function;<br>4. Concepts of weak and proper separation; **Bhawna**<br>1. Theorem 4.1.1;<br>2. Corollary 4.1.3;<br>3. Support function;<br>4. Concepts of weak and proper separation;<br>5. Statement and illustration of Theorem 4.1
- **Bhawna**<br>1. Theorem 4.1.1;<br>2. Corollary 4.1.3;<br>3. Support function;<br>4. Concepts of weak and proper separation;<br>5. Statement and illustration of Theorem 4.1.1. 5. Statement and the University of Theorem 4.1.1;<br>1. Statement and illustration:<br>1. Support function:<br>1. Statement and illustration of Theorem 4.1.1. 1. Theorem 4.1.1;<br>2. Corollary 4.1.3;<br>3. Support function;<br>4. Concepts of weak and proper separation;<br>5. Statement and illustration of Theorem 4.1.1.<br>Note: Provide figures wherever possible for clarity

# Ruhi Sharma

- 1. Article 4.2(a)
- 2. Lemma 4.2.1;
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- **Ruhi Sharma**<br>1. Article 4.2(a)<br>2. Lemma 4.2.1;<br>3. Remark 4.2.2;<br>4. Proposition 4.2.3; **1.** Article 4.2(a)<br> **Ruhi Sharma**<br> **2.** Lemma 4.2.1;<br> **3. Remark 4.2.2;**<br> **4. Proposition 4.2.3;**

1. Article 4.2(a)<br>2. Lemma 4.2.1;<br>3. Remark 4.2.2;<br>4. Proposition 4.2.3;<br>*Note*: Provide figures wherever possible for clarity

# Rupleen Kaur Ahuja

- 1. Article 4.2(b);
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- **Rupleen Kaur Ahuj**<br>1. Article 4.2(b);<br>2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets; **Rupleen Kaur Ahuj**<br>1. Article 4.2(b);<br>2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets;<br>5. Proposition 4.2.7; **4. Anticle 4.2(b);<br>2. Theorem 4.2.4;<br>2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets;<br>5. Proposition 4.2.7; Rupleen Kaur Ahuja**<br>1. Article 4.2(b);<br>2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets;<br>5. Proposition 4.2.7;
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2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets;<br>5. Proposition 4.2.7;<br>Note: Provide figures wherever possible for clarity

# Ajit Kumar

- 1. Article 4.2(b);
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- **Ajit Kumar**<br>1. Article 4.2(b);<br>2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets; **Ajit Kumar**<br>1. Article 4.2(b);<br>2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets;<br>5. Proposition 4.2.7; 4. Article 4.2(b);<br>
2. Theorem 4.2.4;<br>
3. Corollary 4.2.5;<br>
4. Definition 4.2.6 of polyhedral sets;<br>
5. Proposition 4.2.7; **Ajit Kumar**<br>1. Article 4.2(b);<br>2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets;<br>5. Proposition 4.2.7;
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2. Theorem 4.2.4;<br>3. Corollary 4.2.5;<br>4. Definition 4.2.6 of polyhedral sets;<br>5. Proposition 4.2.7;<br>Note: Provide figures wherever possible for clarity

- **Anant Singh<br>ce the concept in Article 5.1);** 1. Article 5.2 (After I introduce the concept in Article 5.1); **Anant Singh**<br>1. Article 5.2 (After I introduce the concept in Ar<br>2. Proposition 5.2.1;<br>3. Definition 5.2.3 of normal cone;<br>4. Proposition 5.2.4; **Anant Singh**<br> **3.** Article 5.2 (After I introduce the concept in Article 5.1);<br>
2. Proposition 5.2.1;<br>
3. Definition 5.2.3 of normal cone;<br>
4. Proposition 5.2.4;<br>
5. Corollary 5.2.5; **Anant Singh**<br>
1. Article 5.2 (After I introduce the concept in Ar<br>
2. Proposition 5.2.1;<br>
3. Definition 5.2.3 of normal cone;<br>
4. Proposition 5.2.4;<br>
5. Corollary 5.2.5;<br>
6. Examples 5.2.6(a) and (b). **Anant Singh**<br>1. Article 5.2 (After I introduce the concept in<br>2. Proposition 5.2.1;<br>3. Definition 5.2.3 of normal cone;<br>4. Proposition 5.2.4;<br>5. Corollary 5.2.5;<br>6. Examples 5.2.6(a) and (b). **Anant Singh**<br>1. Article 5.2 (After I introduce the concept in Article 5.1);<br>2. Proposition 5.2.1;<br>3. Definition 5.2.3 of normal cone;<br>4. Proposition 5.2.4;<br>5. Corollary 5.2.5;<br>6. Examples 5.2.6(a) and (b).
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3. Definition 5.2.3 of normal cone;<br>4. Proposition 5.2.4;<br>5. Corollary 5.2.5;<br>6. Examples 5.2.6(a) and (b).<br>Note: Provide figures wherever possible for clarity

- **Jitendra Singh<br>ne before Proposition 5.3.1;** 1. Properties of tangent cone before Proposition 5.3.1; **1. Properties of tangent cone before Proposition 5.3.1**<br>2. Proposition 5.3.1 (i)-(iv);<br>3. Proposition 5.3.3. **Solution:**<br>3. Properties of tangent cone before Proposition<br>3. Proposition 5.3.1 (i)-(iv);<br>3. Proposition 5.3.3. Solution of the Surfally of the Proposition 5.3.1;<br>
2. Proposition 5.3.1 (i)-(iv);<br>
3. Proposition 5.3.3.<br>
Note: Provide figures wherever possible for clarity
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# Rohit Nageshwar

- 1. Definition 1.1.1;
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- **Rohit Nageshwar**<br>1. Definition 1.1.1;<br>2. Proposition 1.1.2;<br>3. Definitions 1.1.3-1.1.5;<br>4. Proposition 1.1.6; **Rohit Nageshwar**<br>1. Definition 1.1.1;<br>2. Proposition 1.1.2;<br>3. Definitions 1.1.3-1.1.5;<br>4. Proposition 1.1.6;<br>5. Theorem 1.1.8; **Rohit Nageshwar**<br> **1. Definition 1.1.1;**<br> **2. Proposition 1.1.2;**<br> **3. Definitions 1.1.3-1.1.5;**<br> **4. Proposition 1.1.6;**<br> **5. Theorem 1.1.8;**<br> **6. Proposition 1.1.9. Rohit Nageshwar**<br>1. Definition 1.1.1;<br>2. Proposition 1.1.2;<br>3. Definitions 1.1.3-1.1.5;<br>4. Proposition 1.1.6;<br>5. Theorem 1.1.8;<br>6. Proposition 1.1.9. **Rohit Nageshwar**<br> **1.** Definition 1.1.1;<br> **2.** Proposition 1.1.2;<br> **3.** Definitions 1.1.3-1.1.5;<br> **4.** Proposition 1.1.6;<br> **5.** Theorem 1.1.8;<br> **6.** Proposition 1.1.9.
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3. Definitions 1.1.3-1.1.5;<br>4. Proposition 1.1.6;<br>5. Theorem 1.1.8;<br>6. Proposition 1.1.9.<br>Note: Provide figures wherever possible for clarity

# **Himanshu Bhatt**<br>ntinuity:

- 1. Proposition 1.2.1;
- **Himanshu Bhatt**<br>1. Proposition 1.2.1;<br>2. Notion of lower semicontinuity;<br>3. Proposition 1.2.2;<br>4. Definitions 1.2.3-1.2.4; **Himanshu Bhatt**<br>1. Proposition 1.2.1;<br>2. Notion of lower semicontinuity;<br>3. Proposition 1.2.2;<br>4. Definitions 1.2.3-1.2.4;<br>5. Proposition 1.2.5; **Himanshu Bhatt**<br>1. Proposition 1.2.1;<br>2. Notion of lower semicontinuity;<br>3. Proposition 1.2.2;<br>4. Definitions 1.2.3-1.2.4;<br>5. Proposition 1.2.5;<br>6. Proposition 1.2.6; **Himanshu Bhatt**<br>1. Proposition 1.2.1;<br>2. Notion of lower semicontinuity;<br>3. Proposition 1.2.2;<br>4. Definitions 1.2.3-1.2.4;<br>5. Proposition 1.2.5;<br>6. Proposition 1.2.6;<br>7. Notation 1.2.7. **Himanshu Bhatt**<br>1. Proposition 1.2.1;<br>2. Notion of lower semicontinuity;<br>3. Proposition 1.2.2;<br>4. Definitions 1.2.3-1.2.4;<br>5. Proposition 1.2.5;<br>6. Proposition 1.2.6;<br>7. Notation 1.2.7. THIMATISHE DITATE:<br>1. Proposition 1.2.1;<br>2. Notion of lower semicontinuity;<br>3. Proposition 1.2.2;<br>4. Definitions 1.2.3-1.2.4;<br>5. Proposition 1.2.5;<br>6. Proposition 1.2.6;<br>7. Notation 1.2.7.
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4. Definitions 1.2.3-1.2.4;<br>5. Proposition 1.2.5;<br>6. Proposition 1.2.6;<br>7. Notation 1.2.7.<br>*Note*: Provide figures wherever possible for clarity

## Mohit

- 1. Article 1.3(a);
- 
- **Mohit**<br>1. Article 1.3(a);<br>2. Article 1.3(b);<br>3. Article 1.3(c);<br>4. Article 1.3(d).
- **Mohit**<br>1. Article 1.3(a);<br>2. Article 1.3(b);<br>3. Article 1.3(c);<br>4. Article 1.3(d). **Mohit**<br>1. Article 1.3(a);<br>2. Article 1.3(b);<br>3. Article 1.3(c);<br>4. Article 1.3(d).

1. Article 1.3(a);<br>2. Article 1.3(b);<br>3. Article 1.3(c);<br>4. Article 1.3(d).<br>Note: Provide figures wherever possible for clarity

# Gurudatt Rao

- 1. Article 1.3(g);
- 
- **Gurudatt Rao**<br>
1. Article 1.3(g);<br>
2. Theorem 1.3.1;<br>
3. Proposition 2.1.1;<br>
4. Proposition 2.1.2;
- **Gurudatt Rao**<br>1. Article 1.3(g);<br>2. Theorem 1.3.1;<br>3. Proposition 2.1.1;<br>4. Proposition 2.1.2;<br>5. Proposition 2.1.5;
- **Gurudatt Rao**<br>
1. Article 1.3(g);<br>
2. Theorem 1.3.1;<br>
3. Proposition 2.1.1;<br>
4. Proposition 2.1.2;<br>
5. Proposition 2.1.5;<br>
6. Proposition 2.1.6.
- **Gurudatt Rao**<br>1. Article 1.3(g);<br>2. Theorem 1.3.1;<br>3. Proposition 2.1.1;<br>4. Proposition 2.1.2;<br>5. Proposition 2.1.5;<br>6. Proposition 2.1.6. **Gurudatt Rao**<br>1. Article 1.3(g);<br>2. Theorem 1.3.1;<br>3. Proposition 2.1.1;<br>4. Proposition 2.1.2;<br>5. Proposition 2.1.5;<br>6. Proposition 2.1.6.

3. Proposition 2.1.1;<br>4. Proposition 2.1.2;<br>5. Proposition 2.1.5;<br>6. Proposition 2.1.6.<br>Note: Provide figures wherever possible for clarity

# Majhar Alam

- 1. Definition 2.3.1;
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- **Majhar Alam**<br>1. Definition 2.3.1;<br>2. Proposition 2.3.2;<br>3. Remark 2.3.3;<br>4. Properties before Remark 2.3.4; **Majhar Alam**<br>1. Definition 2.3.1;<br>2. Proposition 2.3.2;<br>3. Remark 2.3.3;<br>4. Properties before Remark 2.3.4;<br>5. Remark 2.3.4; Ma**jhar Alam**<br>1. Definition 2.3.1;<br>2. Proposition 2.3.2;<br>3. Remark 2.3.3;<br>4. Properties before Remark 2.3.4;<br>5. Remark 2.3.4;<br>6. Example 2.3.5. Ma**jhar Alam**<br>1. Definition 2.3.1;<br>2. Proposition 2.3.2;<br>3. Remark 2.3.3;<br>4. Properties before Remark 2.3.4;<br>5. Remark 2.3.4;<br>6. Example 2.3.5. Majhar Alam<br>
1. Definition 2.3.1;<br>
2. Proposition 2.3.2;<br>
3. Remark 2.3.3;<br>
4. Properties before Remark 2.3.4;<br>
5. Remark 2.3.4;<br>
6. Example 2.3.5.
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3. Remark 2.3.3;<br>4. Properties before Remark 2.3.4;<br>5. Remark 2.3.4;<br>6. Example 2.3.5.<br>Note: Provide figures wherever possible for clarity

## Monika

- 1. Lemma 3.1.1;
- **Monika**<br>1. Lemma 3.1.1;<br>2. Theorem 3.1.2;<br>3. Remark 3.1.3. **Monika**<br>1. Lemma 3.1.1;<br>2. Theorem 3.1.2;<br>3. Remark 3.1.3.
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**Monika**<br>1. Lemma 3.1.1;<br>2. Theorem 3.1.2;<br>3. Remark 3.1.3.<br>*Note*: Provide figures wherever possible for clarity

## Sachin Kumar **Sachin Kumar**<br>1. Theorem 3.1.5;<br>2. Theorem 4.1.1;<br>3. Definition 4.1.3;<br>4. Theorem 4.1.4. **Sachin Kumar**<br>1. Theorem 3.1.5;<br>2. Theorem 4.1.1;<br>3. Definition 4.1.3;<br>4. Theorem 4.1.4. **Sachin Kumar**<br>1. Theorem 3.1.5;<br>2. Theorem 4.1.1;<br>3. Definition 4.1.3;<br>4. Theorem 4.1.4.

- 1. Theorem 3.1.5;
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Sachin Kumar<br>1. Theorem 4.1.1;<br>2. Theorem 4.1.1;<br>4. Theorem 4.1.4.<br>Note: Provide figures wherever possible for clarity