Name : Rahul Mansotra

Roll No : P2108

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Q 17 Let F be a mapping of a complete metric space into itself. If x is a fixed point of F^m (for some m), does it follow that x is a fixed point of F?

Sol. Neednot be

Example : Let $F : [0,1] \longrightarrow [0,1]$ defined by

$$F(x) = \begin{cases} 2x & , x \in [0, \frac{1}{2}] \\ 2(1-x), & x \in [\frac{1}{2}, 1] \end{cases}$$

Note: $x = 0, \frac{2}{3}$ are the fixed pts. of F

Now, $F^2 : [0,1] \longrightarrow [0,1]$

$$F^2(x) = \begin{cases} 4x & , x \in [0, \frac{1}{4}] \\ 2-4x, & x \in [\frac{1}{4}, \frac{1}{2}] \\ 4x-2, & x \in [\frac{1}{2}, \frac{3}{4}] \\ 4-4x & , x \in [\frac{3}{4}, 1] \end{cases}$$

Note: $x = 0, \frac{2}{3}, \frac{2}{5}, \frac{4}{5}$ are the fixed pts. of F^2 and $[0,1]$ is complete being closed subset of complete metric space.

Note: $x = \frac{4}{5}$ is not a fixed point of F still $x = \frac{4}{5}$ is a fixed point of F^2.

Q 18 Let $f : [0,b] \times \mathbb{R} \to \mathbb{R}$. Prove that if f and $\frac{\partial f}{\partial t}$ are cts (t being the second argument of f) and $\frac{\partial f}{\partial t}$ are bounded, then the I.V.P $x'(s) = f(s, x(s))$, $x(0) = \beta$ has a unique soln on $[0,b]$.

Sol. The Initial Value Problem (I.V.P) $x'(s) = f(s, x(s))$ $x(0) = \beta$ is equivalent to

$Ax = x$, $Ax(s) = \int_0^s f(s, x(s))\,ds + \beta$

Since $\frac{\partial f}{\partial t}$ is bdd $\Rightarrow \exists\, M \in \mathbb{R}^+$ s.t $\left|\frac{\partial f}{\partial t}\right| \le M$

Define $\phi_s : \mathbb{R} \to \mathbb{R}$, $\phi_s(t) = f(s,t)$

$\Rightarrow \phi_s$ is cts. as f is cts.

Note: for any $t_1, t_2 \in \mathbb{R}$, by Mean value Theorem

$$\phi_s(t_2) - \phi_2(t_1) = \left(\frac{\partial \phi_s}{\partial t}(t_3)\right)(t_2 - t_1) \text{ for some } t_3 \in (t_1, t_2)$$

i.e $f(s,t_2) - f(s,t_1) = \frac{\partial f}{\partial t}(s,t_3)(t_2 - t_1) \to (1)$

Define a new function $\|\cdot\|_W$ on $C[0,b]$ as

$$\|x\|_W = \sup_{s \in [0,b]} |x(s)|\, e^{-2Ms}$$ clearly $\|\cdot\|_W$ is a norm on $C[0,b]$

Note: $e^{-2Mb}\|x\| \le \|x\|_W \le \|x\|$

$\Rightarrow (C[0,b], \|\cdot\|_W)$ is complete as $(C[0,b], \|\cdot\|)$ is complete.

In order to apply contraction Mapping Theorem, we need to check that

1) $A : C[0,b] \to C[0,b]$
2) $(C[0,b], \|\cdot\|_W)$ is complete
3) A is contraction.

clearly (1) & (2) follows

<u>For (3)</u>, $|Au(p) - Av(p)| = \left|\int_0^p (f(s,u(s)) - f(s,v(s)))\,ds\right|$

$$\le \int_0^p |f(s,u(s)) - f(s,v(s))|\,ds$$

$$= \int_0^p \left|\frac{\partial f}{\partial t}(s,t_3)\,(u(s) - v(s))\right|\,ds$$

for some $t_3 \in (u(s), v(s))$ (by (1))

$$|Au(p) - Av(p)| \leq M \int_0^p |u(s) - v(s)|\, ds$$

$$= M \int_0^p e^{2Ms} e^{-2Ms} |u(s) - v(s)|\, ds$$

$$\leq M \|u - v\| \int_0^p e^{2Ms}\, ds$$

$$= M \|u - v\| \left.\frac{e^{2Ms}}{2M}\right|_0^p$$

$$= \frac{1}{2} \|u - v\| (e^{2Mp} - 1)$$

$$\leq \frac{1}{2} \|u - v\| e^{2Mp}$$

$$e^{-2Mp} |Au(p) - Av(p)| \leq \frac{1}{2} \|u - v\|$$

$$\|Au - Av\| \leq \frac{1}{2} \|u - v\|$$

$\Rightarrow$ A is a contraction from $(C[0,b], \|\cdot\|_w)$ to $(C[0,b], \|\cdot\|_w)$

by Contraction Mapping Theorem

$\exists\, x \in C[0,b]$ (unique) s.t $Ax = x$

i.e $x'(s) = f(s, x(s))$ has unique soln on $[0,b]$.

Q19 Prove that if $F: \mathbb{R}^n \to \mathbb{R}^n$ is a contraction, then $F + I$ is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Sol. Clearly F is cts as F is contraction on $\mathbb{R}^n$

$\Rightarrow \exists\, \lambda \in (0,1)$ s.t $\|Fx - Fy\| \leq \lambda \|x - y\|$

for any $\epsilon > 0$, $\exists\, \delta = \frac{\epsilon}{\lambda}$ s.t whenever $\|x - y\| < \delta$

$$\|Fx - Fy\| \leq \lambda \|x - y\| < \lambda\delta = \epsilon$$

Also I is cts

$\Rightarrow$ I + F is cts

I+F is 1-1 :

Suppose $(F+I)(x_1) = (F+I)(x_2)$

$\Rightarrow F(x_1) + I(x_1) = F(x_2) + I(x_2)$

$\Rightarrow F(x_1) - F(x_2) = x_2 - x_1$

On the contrary suppose $x_1 \neq x_2$, then

$\|x_1 - x_2\| = \|F(x_1) - F(x_2)\| \leq \lambda \|x_1 - x_2\|$

$< \|x_1 - x_2\|$, $\lambda \in (0,1)$

which is absurd.

$\Rightarrow x_1 = x_2$ $\Rightarrow$ I+F is 1-1

I+F is onto : Let $v \in \mathbb{R}^n$

Note : $v-f$ is a contraction on $\mathbb{R}^n$

$\because \|(v-f)(x) - (v-f)(y)\| = \|v - f(x) - v + f(y)\|$

$= \|f(x) - f(y)\| \leq \lambda \|x-y\|$, $\lambda \in (0,1)$

as f is contraction on $\mathbb{R}^n$

Note : $\mathbb{R}^n$ is complete and $v-f : \mathbb{R}^n \to \mathbb{R}^n$

by Contraction Mapping Theorem

$(v-f)(x) = x$

$\Rightarrow v = x + f(x)$

$\Rightarrow v = (f+I)(x)$ as v was arb.

$\Rightarrow$ I+F is onto

$(I+F)^{-1}$ is continuous ; It is equivalent to

I+F is open map.

Let U be an open set in $\mathbb{R}^n$

$(I+F)(U) = \bigcup_{v \in U} (F(v) + U)$

Since $F: X \to X$, X is Topological Vector Space defined by $F(x) = x + a$, for some $a \in X$ (fixed) $\Big\} \to$ (II)

Then F is a homeomorphism

$\Rightarrow F(v) + U$, $\forall v \in U$ is open

$\Rightarrow \bigcup_{v \in U} (F(v) + U)$ is open

$\Rightarrow (I + F)(U)$ is open

Thus, $I + F$ is cts, 1-1, onto & $(I+F)^{-1}$ is cts

Hence, $I + F$ is homeomorphism on $\mathbb{R}^n$.

<u>Note</u>: if $\lambda = 0$, then F is constant

say $F(x) = a$, for some $a \in \mathbb{R}^n$ (fixed)

clearly F is cts as $\|F(x) - F(y)\| = 0$

$\Rightarrow \|F(x) - F(y)\| < \|x - y\|$, $x \neq y$

for any $\epsilon > 0$, $\exists \delta = \epsilon$ s.t. whenever

$\|x - y\| < \delta \Rightarrow \|F(x) - F(y)\| \leq \|x - y\| < \epsilon$

i.e $\|F(x) - F(y)\| < \epsilon$

Also I is cts

$\Rightarrow I + F$ is cts

<u>$I + F$ is 1-1</u>: Suppose $(I+F)(x_1) = (I+F)(x_2)$

$\Rightarrow x_1 + F(x_1) = x_2 + F(x_2)$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

$\Rightarrow I + F$ is 1-1

<u>$I + F$ is onto</u>: Let $v \in \mathbb{R}^n$.

clearly, $\exists\, v - a \in \mathbb{R}^n$ s.t $(I+F)(v-a)$

$= v - a + F(v - a)$

$= v - a + a = v$

$\Rightarrow$ $I + F$ is onto

$(I+F)^{-1}$ is cts : as $I+F$ is open is equivalent to $(I+F)^{-1}$ is cts.

Let U be an open set in $\mathbb{R}^n$

$(I+F)(U) = a + U$

by (II) $(I+F)(U)$ is open

Thus, $I+F$ is cts, 1-1, onto & open

Hence $I+F$ is homeomorphism on $\mathbb{R}^n$.