6th July

$\mathcal{A} \to$ algebra, set of complex (or real) fns on a set E st.

(i) $f+g \in \mathcal{A}$, if $f, g \in \mathcal{A}$

(ii) $fg \in \mathcal{A}$, " "

(iii) $cf \in \mathcal{A}$ if $f \in \mathcal{A}$, $c \in \mathbb{C}$ (or $c \in \mathbb{R}$)

$C[a,b]$ is an algebra

$P[a,b]$: all poly. defined on $[a,b]$

$L^2(\mathbb{R})$,
$\rightarrow$ not an algebra.

An algebra $\mathcal{A}$ is said to be uniformly closed iff $\{f_n\} \subseteq \mathcal{A} \longrightarrow f$ unif on E
$\Rightarrow f \in \mathcal{A}$

Weierstrass Approx Theorem.

$C[a,b]$ is the uniform closure of $P[a,b]$.

Theorem: Let $\mathcal{A}$ be an algebra of bounded functions. Let $\mathcal{B}$ be the uniform closure of $\mathcal{A}$. Then $\mathcal{B}$ is also an algebra.

Pf:

<u>Defn</u>: Let $\mathcal{A}$ be an family of functions defined on a set E. We say $\mathcal{A}$ separates points of E if for every $x_1, x_2 \in E$ $x_1 \neq x_2$;

$\exists f \in \mathcal{A}$ s.t.

$$f(x_1) \neq f(x_2).$$

Eg: $C[a,b] \longrightarrow$ separate points of $[a,b]$?

L05

$\therefore$ $x_1, x_2 \in [a,b]$

$x_1 \neq x_2$.

Take $f(x) = x$.

Then $f(x_1) \neq f(x_2)$.

$A \equiv$ set of all even polyn

$= \{ p(x) = a_0 + a_1 x^2 + a_2 x^4 + \cdots$

$a_n x^{2n}$

$a_i \in \mathbb{C}$. $\}$

$x \in [-2, 2]$.

$p(-t) = p(t) \quad \forall t \in [0, 2]$.

<u>Theorem</u>: Suppose $\mathcal{A}$ is an algebra of functions on a set E, $\mathcal{A}$ separating pts on E, and $\mathcal{A}$ vanishing at no point of E.
Suppose x_1, x_2 are two distinct pts of E and c_1, c_2 are constant scalar.
Then $\mathcal{A}$ contains a func f st $f(x_1) = c_1$, $f(x_2) = c_2$.

<u>$\mathcal{A}$ vanishing at no pt of E</u>: To each $x \in E$, $\exists$ $g \in \mathcal{A}$, st
$$g(x) \neq 0 .$$

Pf. Let $x_1, x_2 \in E$
$x_1 \neq x_2$.

$\mathcal{A}$ separates pts $\Rightarrow \exists\, g \in \mathcal{A}$
st $g(x_1) \neq g(x_2)$. — (1)

and $\mathcal{A}$ vanishes at no pt
$\Rightarrow \exists\, h \in \mathcal{A}$ st — (2)
$h(x_1) \neq 0$

Set
$$u = g + \lambda h.$$
where λ is a scalar
chosen as follows:
If $g(x_1) \neq 0$, choose $\lambda = 0$.

If $g(x_1) = 0$, then $g(x_2) \neq 0$

choose λ st
$\lambda [h(x_1) - h(x_2)] \neq g(x_2)$.

Then $u \in \mathcal{A}$
$$u(x_1) - u(x_2)$$
$$= g(x_1) - g(x_2) + \lambda [h(x_1) - h(x_2)] \quad \text{— } 2$$
$\neq 0$ by our choice of λ.

and $u(x_1) \neq 0$. — 3

Set

$$\alpha = u^2(x_1) - u(x_1)u(x_2)$$

Then $\alpha \neq 0$. by (2) & (3).

Set

$$f_1 = \frac{u^2 - u(x_2)u}{\alpha}.$$

$\therefore f_1 \in \mathcal{a}$.

$$f_1(x_1) = \frac{u^2(x_1) - u(x_2)u(x_1)}{\alpha}.$$

$$= 1.$$

$$f_1(x_2) = 0$$

Illy we can find
$f_2 \in \mathcal{A}$ st
$f_2(x_2) = 1, \quad f_2(x_1) = 0.$

Choose
$$f = c_1 f_1 + c_2 f_2.$$

$\in \mathcal{A}$
and works.

(X, d), $K \subseteq X$, compact

$\mathcal{A} = \{ f : K \to \mathbb{R},\ f \text{ cts} \}$

$\mathcal{A}$ is algebra over $\mathbb{R}$?

Stone's Generalization of Weierstrass Theorem (Real).

<u>Theorem</u>: Let $\mathcal{A}$ be the algebra of real continuous functions on a compact set K. If

(i) $\mathcal{A}$ separates pts of K.

(ii) $\mathcal{A}$ vanishes at no pt of K,

then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ is $C(K)$ in space $\mathbb{R}$.

of all cts real valued
fncs on K.

Pf Step I

Claim: If $f \in \mathcal{B}$,
then $|f| \in \mathcal{B}$.

where $\mathcal{B} \equiv$ unif closure of $\mathcal{A}$.

Fix $f \in \mathcal{B}$.

Let $a = \sup_{x \in K} |f(x)|$

a is finite ($\because f$ is the uniform limit of cts functions, and K is compact)

Let $\varepsilon > 0$ be given.

By Weierstrass app Th. the following holds:

Result: For every interval $[-c, c]$, there is a sequence of polynomials P_n, such that $\boxed{P_n(0) = 0}$ and

$$\lim_{n \to \infty} P_n(x) = |x|$$

uniformly on $[-c, c]$.

(By W. app. Th, $\exists \{Q_n\}$ st $Q_n(x) \longrightarrow |x|$. unif.

write $P_n(x) = Q_n(x) - Q_n(0)$

$\because Q_n(0) \longrightarrow 0$ as $n \to \infty$

$\{P_n\}$ works). Using this Result.

$\exists$ real nos $c_1, c_2, \ldots c_{j_n}$ s.t.

$$\underbrace{\left| \sum_{i=1}^{j_n} c_i y^i - |y| \right|}_{P_n(y)} < \varepsilon \quad \forall y \in [-a, a]. \quad — (1)$$

Set $g = \sum_{i=1}^{j_n} c_i f^i \in \mathcal{B}$

$\therefore$

$$\left| g(x) - |f(x)| \right|$$

$$= \left| \sum_{i=1}^{j_n} c_i f^i(x) - |f(x)| \right|$$

$< \varepsilon$ (by ① and the fact that

$-a < f(x) < a$).

for all $x \in K$.

$$\Rightarrow |f| \in \mathcal{B} = \overline{\mathcal{A}}^u.$$

$\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$.

Then, by previous dis,

$\exists\ g_n$ s.t.

$$|g_n(x) - |f(x)|| < \frac{1}{n} \quad \forall x \in K.$$

Thus $\{g_n\}$ is a seq in $\mathcal{B}$ converging unif to $|f|$ on K.

But $\mathcal{B}$ is unif closed.

$\Rightarrow |f| \in \mathcal{B}$.