R-07
Convex and Nonsmooth Analysis

Relative interior operation does not distinguish ri$C$, C and cl$C$

Theorem If C is a nonempty convex set in $\mathbb{R}^n$ then

$$\mathrm{ri}(\mathrm{ri}C) = \mathrm{ri}C = \mathrm{ri}(\mathrm{cl}C).$$

Proof It is enough to prove $\mathrm{ri}(\mathrm{cl}C) = \mathrm{ri}C$.

Since $C \subseteq \mathrm{cl}C$ and $\mathrm{aff}C = \mathrm{aff}(\mathrm{cl}C)$ it follows that $\mathrm{ri}C \subseteq \mathrm{ri}(\mathrm{cl}C)$.

Let $y \in \mathrm{ri}(\mathrm{cl}C)$. Since $\mathrm{ri}C \neq \emptyset$ we can find $x' \in \mathrm{ri}C$.

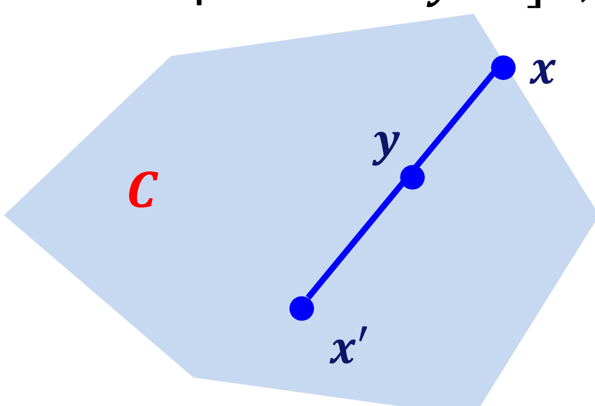
If $y = x'$ then $y \in \mathrm{ri}C$ and we are done. Let $y \neq x'$.

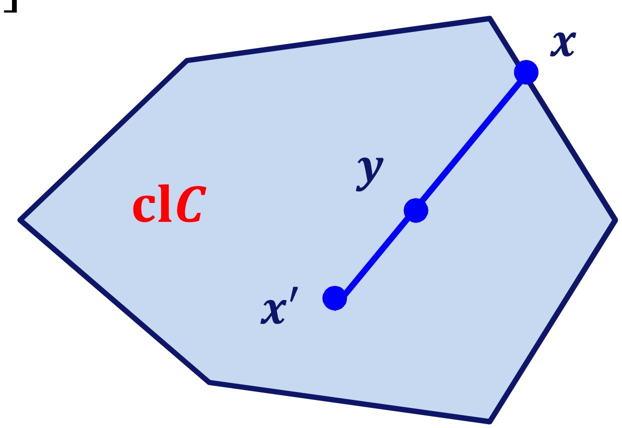
Since $x' \in \mathrm{ri}C \subseteq \mathrm{cl}C$ and $y \in \mathrm{ri}(\mathrm{cl}C)$ by the stretching process there exists $x \in \mathrm{cl}C$ such that

$$[x, x'] \subseteq \mathrm{cl}C \text{ and } y \in \,]x, x'[\,.$$

Now $x' \in \mathrm{ri}C$ and $x \in \mathrm{cl}C$ hence by absorption property $]x, x'] \subseteq \mathrm{ri}C$.

This implies that $y \in \,]x, x'[\, \subseteq \,]x, x'] \subseteq \mathrm{ri}C$.

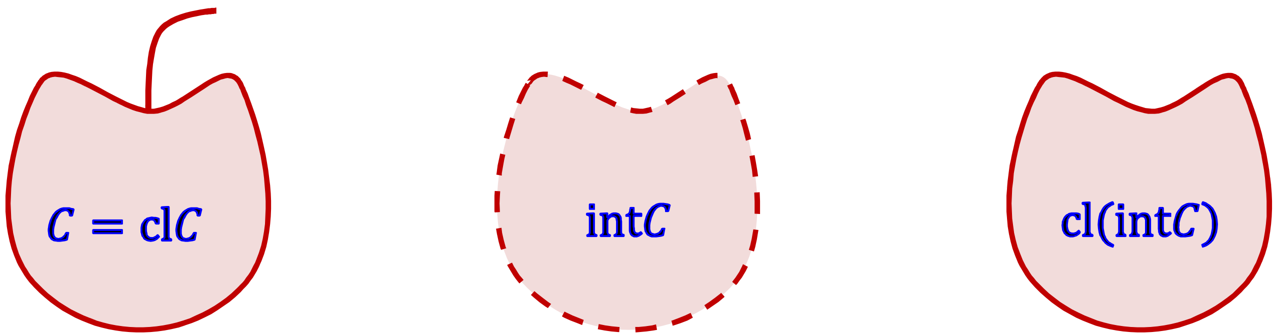




What if C is not convex?

Theorem If C is a nonempty convex set in $\mathbb{R}^n$ then

$$\mathrm{cl}(\mathrm{ri}C) = \mathrm{cl}C = \mathrm{cl}(\mathrm{cl}C).$$



Theorem If C is a nonempty convex set in $\mathbb{R}^n$ then

$$\mathrm{ri}(\mathrm{ri}C) = \mathrm{ri}C = \mathrm{ri}(\mathrm{cl}C).$$

$\boldsymbol{C} = \mathbb{Q}$,

$\mathrm{int}C = \emptyset$,

$\mathrm{cl}\boldsymbol{C} = \mathbb{R} = \mathrm{int}(\mathrm{cl}\boldsymbol{C})$

Property of Relative Interior

Theorem Let C_1 and C_2 be two convex set in $\mathbb{R}^n$ such that $\mathrm{ri}C_1 \cap \mathrm{ri}C_2 \neq \emptyset$. Then

i) $\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}C_1 \cap \mathrm{cl}C_2$,

ii) $\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}C_1 \cap \mathrm{ri}C_2$.

Proof We observe that $C_1 \cap C_2 \neq \emptyset$ as $\mathrm{ri}C_1 \cap \mathrm{ri}C_2 \neq \emptyset$.

i) Clearly, $C_1 \cap C_2 \subseteq \mathrm{cl}C_1 \cap \mathrm{cl}C_2$, which implies that

$$\mathrm{cl}(C_1 \cap C_2) \subseteq \mathrm{cl}C_1 \cap \mathrm{cl}C_2.$$

Let $x \in \mathrm{cl}C_1 \cap \mathrm{cl}C_2$. Since $\mathrm{ri}C_1 \cap \mathrm{ri}C_2 \neq \emptyset$ we can find $x' \in \mathrm{ri}C_1 \cap \mathrm{ri}C_2$.

If $x = x'$ then $x \in \mathrm{ri}C_1 \cap \mathrm{ri}C_2 \subseteq C_1 \cap C_2 \subseteq \mathrm{cl}(C_1 \cap C_2)$ and we are done.

Let $x \neq x'$. By absorption property

$$]x, x'] \subseteq \mathrm{ri}C_1 \cap \mathrm{ri}C_2.$$

Hence

$$x \in [x, x'] \subseteq \mathrm{cl}(\mathrm{ri}C_1 \cap \mathrm{ri}C_2) \subseteq \mathrm{cl}(C_1 \cap C_2).$$

Hence, $\mathrm{cl}C_1 \cap \mathrm{cl}C_2 \subseteq \mathrm{cl}(C_1 \cap C_2)$.

continued

ii) By part i) applied to the sets $\mathrm{ri}C_1$ and $\mathrm{ri}C_2$.

$$\mathrm{cl}(\mathrm{ri}C_1 \cap \mathrm{ri}C_2) = \mathrm{cl}(\mathrm{ri}C_1) \cap \mathrm{cl}(\mathrm{ri}C_2) = \mathrm{cl}C_1 \cap \mathrm{cl}C_2.$$

Also,

$$\mathrm{cl}C_1 \cap \mathrm{cl}C_2 = \mathrm{cl}(C_1 \cap C_2).$$

Hence,

$$\mathrm{cl}(\mathrm{ri}C_1 \cap \mathrm{ri}C_2) = \mathrm{cl}(C_1 \cap C_2).$$

This implies that

$$\mathrm{ri}(\mathrm{cl}(\mathrm{ri}C_1 \cap \mathrm{ri}C_2)) = \mathrm{ri}(\mathrm{cl}(C_1 \cap C_2)).$$

As $\mathrm{ri}(\mathrm{cl}A) = \mathrm{ri}A$, for any set A we have

$$\mathrm{ri}(\mathrm{ri}C_1 \cap \mathrm{ri}C_2) = \mathrm{ri}(C_1 \cap C_2).$$

As $\mathrm{ri}(\mathrm{ri}C_1 \cap \mathrm{ri}C_2) \subseteq \mathrm{ri}C_1 \cap \mathrm{ri}C_2$ we have

$$\mathrm{ri}(C_1 \cap C_2) \subseteq \mathrm{ri}C_1 \cap \mathrm{ri}C_2.$$

continued

Claim $\mathrm{ri}C_1 \cap \mathrm{ri}C_2 \subseteq \mathrm{ri}(C_1 \cap C_2)$

Let $y \in \mathrm{ri}C_1 \cap \mathrm{ri}C_2$. Let $x' \in \mathrm{ri}(C_1 \cap C_2)$. If $y = x'$ then $y \in \mathrm{ri}(C_1 \cap C_2)$ and we are done. Let $y \neq x'$.

Since $x' \in C_1$ and $y \in \mathrm{ri}C_1$ there exists $u \in C_1$ such that

$$[u, x'] \subseteq C_1 \text{ and } y \in \,]u, x'[\, .$$

Since $x' \in C_2$ and $y \in \mathrm{ri}C_2$ there exists $v \in C_2$ such that

$$[v, x'] \subseteq C_2 \text{ and } y \in \,]v, x'[\, .$$

If $[v, x'] \subseteq [u, x']$ we choose $x = v$ and if $[u, x'] \subseteq [v, x']$ we choose $x = u$.

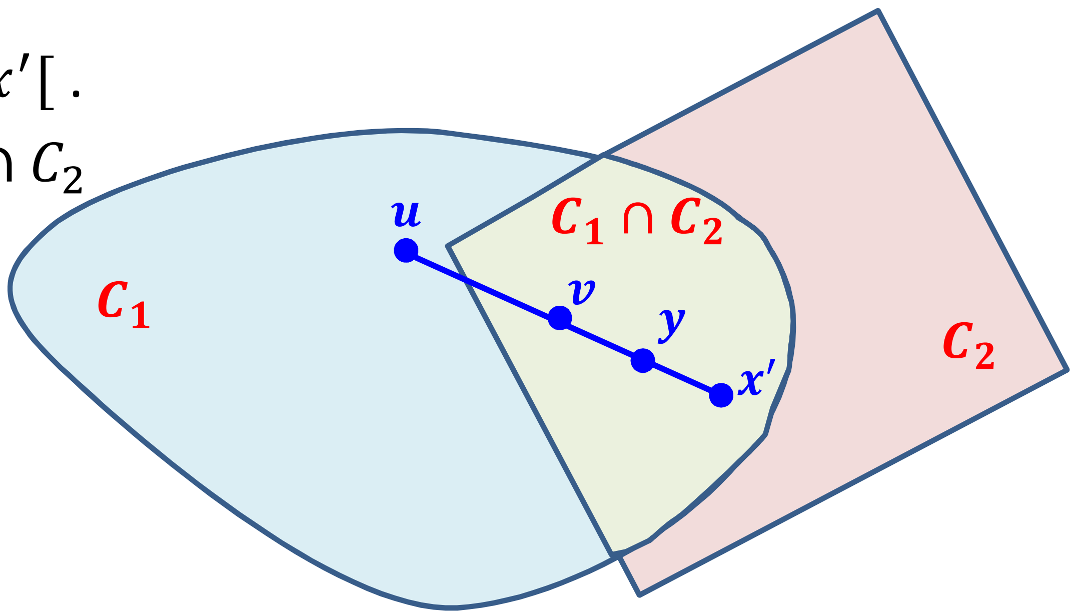
Hence,

$$[x, x'] \subseteq C_1 \cap C_2 \text{ and } y \in \,]x, x'[\, .$$

As $x' \in \mathrm{ri}(C_1 \cap C_2)$ and $x \in C_1 \cap C_2$ by the absorption property on $C_1 \cap C_2$ we have
$]x, x'] \subseteq \mathrm{ri}(C_1 \cap C_2)$.

As $y \in \,]x, x'[$ it follows that

$$y \in \mathrm{ri}(C_1 \cap C_2).$$



What if $\mathbf{ri}\boldsymbol{C_1} \cap \mathbf{ri}\boldsymbol{C_2} = \emptyset$?

Theorem Let C_1 and C_2 be two convex set in $\mathbb{R}^n$ such that $\mathrm{ri}C_1 \cap \mathrm{ri}C_2 \neq \emptyset$.
Then

i) $\mathrm{cl}(C_1 \cap C_2) = \mathrm{cl}C_1 \cap \mathrm{cl}C_2$,

ii) $\mathrm{ri}(C_1 \cap C_2) = \mathrm{ri}C_1 \cap \mathrm{ri}C_2$.

i) $C_1 =]0,1]$ and $C_2 = \{0\}$ be two convex set in $\mathbb{R}$

$\mathrm{ri}C_1 \cap \mathrm{ri}C_2 = \emptyset$ as $\mathrm{ri}C_1 =]0,1[, \mathrm{ri}C_2 = \{0\}$.

$$\mathrm{cl}(C_1 \cap C_2) = \emptyset, \quad \mathrm{cl}C_1 \cap \mathrm{cl}C_2 = \{0\}.$$

ii) $C_1 = [0,1]$ and $C_2 = \{0\}$ be two convex set in $\mathbb{R}$

$\mathrm{ri}C_1 \cap \mathrm{ri}C_2 = \emptyset$ as $\mathrm{ri}C_1 =]0,1[, \mathrm{ri}C_2 = \{0\}$.

$$\mathrm{ri}(C_1 \cap C_2) = \{0\}, \quad \mathrm{ri}C_1 \cap \mathrm{ri}C_2 = \emptyset.$$

Monotonicity again

Lemma If $A \subseteq B$ and $\mathrm{aff}A = \mathrm{aff}B$ then $\mathrm{ri}A \subseteq \mathrm{ri}B$.

Lemma If $A \subseteq B$ and $\mathrm{ri}A \cap \mathrm{ri}B \neq \emptyset$ then $\mathrm{ri}A \subseteq \mathrm{ri}B$.

Proof We have $\mathrm{ri}(A \cap B) = \mathrm{ri}A \cap \mathrm{ri}B$.

As $A \cap B = A$ we have

$$\mathrm{ri}A = \mathrm{ri}(A \cap B) = \mathrm{ri}A \cap \mathrm{ri}B$$

$$\subseteq \mathrm{ri}B.$$

Properties of Relative Interior

Theorem If $C_i, i = 1, 2, \dots, k$ are convex sets in $\mathbb{R}^{n_i}$, $i = 1, 2, \dots, k$ then

$$\text{ri}(C_1 \times C_2 \times \ \dots \times C_k) = (\text{ri}C_1) \times (\text{ri}C_2) \times \cdots \times (\text{ri}C_k).$$

Theorem Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map and C be a nonempty convex set in $\mathbb{R}^n$. Then

$$\text{ri}[A(C)] = A[\text{ri}C].$$

If D is a convex set in $\mathbb{R}^m$ such that $A^{-1}(\text{ri}D) \neq \emptyset$ then

$$\text{ri}[A^{-1}(D)] = A^{-1}(\text{ri}D)$$

where $A^{-1}(D) = \{x \in \mathbb{R}^n : A(x) \in D\}$.

Proof First show that for a set S in $\mathbb{R}^n$

$$A(\text{cl}S) \subseteq \text{cl}(A(S)).$$

Claim $\text{ri}[A(C)] \subseteq A[\text{ri}C]$.

continued

$$A(C) \subseteq A(\mathrm{cl}C) = A(\mathrm{cl}(\mathrm{ri}C))$$
$$\subseteq \mathrm{cl}\big(A(\mathrm{ri}C)\big) \subseteq \mathrm{cl}(A(C)).$$

Taking closure operation

$$\mathrm{cl}\big(A(C)\big) = \mathrm{cl}\big(A(\mathrm{ri}C)\big)$$

which implies

$$\mathrm{ri}\left(\mathrm{cl}\big(A(C)\big)\right) = \mathrm{ri}(\mathrm{cl}(A(\mathrm{ri}C))).$$

Hence

$$\mathrm{ri}\big(A(C)\big) = \mathrm{ri}(A(\mathrm{ri}C)) \subseteq A[\mathrm{ri}C].$$

Claim $A[\mathrm{ri}C] \subseteq \mathrm{ri}[A(C)]$.

Let $w = A(y) \in A[\mathrm{ri}C]$.

Choose $z' = A(x') \in \mathrm{ri}[A(C)]$.

If $z' = w$ then we are done.

Let $z' \neq w$ then $x' \neq y$. Since $x' \in C$, $y \in \mathrm{ri}C$ by stretching process there exists $x \in C$ such that $y \in\,]x, x'[$.

continued

Since A is affine $A(y) \in]A(x), A(x')[=]z, z'[$.
Since $z' \in \mathrm{ri}[A(C)], z \in A(C)$ by absorption property it follows that $]z, z'] \subseteq \mathrm{ri}[A(C)]$ and hence $A(y) \in \mathrm{ri}[A(C)]$.
ii) Do it yourself.

Corollary Let C_1 and C_2 be two convex set in $\mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.
Then

$$\mathrm{ri}(\alpha_1 C_1 + \alpha_2 C_2) = \alpha_1 \mathrm{ri} C_1 + \alpha_2 \mathrm{ri} C_2.$$

Note $\mathrm{ri}(C_1 - C_2) = \mathrm{ri} C_1 - \mathrm{ri} C_2.$

Note $0 \in \mathrm{ri}(C_1 - C_2) \Leftrightarrow 0 \in \mathrm{ri} C_1 - \mathrm{ri} C_2 \Leftrightarrow \mathrm{ri} C_1 \cap \mathrm{ri} C_2 \neq \emptyset.$