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Q.9) $(X, d) \to$ metric space.

$F: X \to X$ s.t

(i) $d(Fx, Fy) < d(x, y) \quad \forall x \neq y$

(ii) For $x \in X$, $\langle F^n x \rangle$ has a cluster point.

then prove that this cluster point is a fixed point of F.

Proof: Let $\alpha_n = d(F^{n+1}x, F^n x)$

$\Rightarrow 0 \leq \alpha_n < \alpha_{n-1} < \alpha_{n-2} \cdots$

$[\, d(Fx, Fy) < d(x, y), \forall x \neq y \,]$

$\langle \alpha_n \rangle \to$ seq of real numbers + bounded below by zero + monotonic decreasing.

then by monoton conv thm

$\langle \alpha_n \rangle \to \alpha$ (say).

Given For $x \in X$, $\langle F^n x \rangle$ has a cluster point say u.

$\Rightarrow \exists$ a subseq $\langle F^{n_k} x \rangle$ of $\langle F^n x \rangle$ s.t

$\langle F^{n_k} x \rangle \to u$.

consider $\alpha_{n_k} = d(F^{n_k+1}x, F^{n_k}x)$

$\langle \alpha_{n_k} \rangle \to \alpha$ $\quad [\langle \alpha_{n_k} \rangle$ is a sub seq of $\langle \alpha_n \rangle]$

$\Rightarrow \alpha = \lim_{k \to \infty} \alpha_{n_k} = \lim_{k \to \infty} d(F^{n_k+1}x, F^{n_k}x)$

$= d(F(\lim_{k \to \infty} F^{n_k}x), \lim F^{n_k} x)$

Spiral

$= d(F(u), u)$ — ①

$[\because \lim_{k\to\infty} F^{n_k}x = u]$

Now, consider

$\alpha_{n_k+1} = d(F^{n_k+2}x, F^{n_k+1}x)$

$\langle \alpha_{n_k+1} \rangle \to \alpha$, $(\langle \alpha_{n_k+1} \rangle$ sub seq. of $\langle \alpha_n \rangle)$

$\Rightarrow \alpha = \lim_{k\to\infty} \alpha_{n_k+1}$

$= \lim_{k\to\infty} d(F^{n_k+2}x, F^{n_k+1}x)$

$= d(F^2(u), F(u))$ — ②

by uniqueness of the limit

① = ②

$\Rightarrow$

$d(F(u), u) = d(F^2(u), F(u))$ — ③

if $F(u) \neq u$

then by (i) $d(F^2(u), F(u)) < d(F(u), u)$

by ③ $d(F(u), u) < d(F(u), u)$ ✗

$\Rightarrow F(u) = u$

$\Rightarrow u$ is a fixed point of F.

Example:

$f: \mathbb{R} \to \mathbb{R}$ given by , $\mathbb{R} \to$ Banach space

$$f(x) = \frac{x + \sqrt{x^2+1}}{2}$$

$$|f(x) - f(y)| = \left| \frac{x + \sqrt{x^2+1}}{2} - \frac{y + \sqrt{y^2+1}}{2} \right|$$

$$= \left| \frac{x - y}{2} + \frac{\sqrt{x^2+1} - \sqrt{y^2+1}}{2} \right|$$

$$= \left| \frac{x - y}{2} + \frac{x^2+1 - y^2 - 1}{2\left[\sqrt{x^2+1} + \sqrt{y^2+1}\right]} \right|$$

$$= \left| \frac{x - y}{2} + \frac{(x-y)(x+y)}{2\left[\sqrt{x^2+1} + \sqrt{y^2+1}\right]} \right| \quad -(1)$$

$x \leq |x| \quad \Rightarrow \quad x \leq \sqrt{x^2}$

$\Rightarrow \quad x < \sqrt{x^2+1} \quad -(2)$

simill. $\quad y < \sqrt{y^2+1} \quad -(3)$

From (2) & (3)

$$x + y < \sqrt{x^2+1} + \sqrt{y^2+1}$$

$$\Rightarrow \frac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}} < 1 \quad -(4)$$

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From ①

$$|f(x) - f(y)| = \left| \frac{x-y}{2} + \frac{(x-y)(x+y)}{2[\sqrt{x^2+1} + \sqrt{y^2+1}]} \right|$$

$$\leq \frac{|x-y|}{2} + \frac{|x-y||x+y|}{2[\sqrt{x^2+1} + \sqrt{(y^2+1)}]}$$

$$< \frac{|x-y|}{2} + \frac{|x-y|}{2} = |x-y|$$

$\Rightarrow \quad |f(x) - f(y)| < |x-y| \quad \forall x \neq y \in \mathbb{R}$

thus f satisfied given condⁿ (i) in que ⑨.

But

$f(x) = x$

$\Rightarrow \quad \dfrac{x + \sqrt{x^2+1}}{2} = x$

$\Rightarrow \quad x + \sqrt{x^2+1} = 2x$

$\Rightarrow \quad \sqrt{x^2+1} = x$

$\Rightarrow \quad x^2+1 = x^2 \Rightarrow 1 = 0$ ✗

$\Rightarrow$ f has no fixed point. even $\mathbb{R}$ space is Banach space.