Book: Analysis for applied Mathematics by Ward Cheney

Exercise : 3.1 | Problems : 15 & 17 | Page No. : 120

P-15 Let y_0 be a point in a normed linear space Y. Define $f: \mathbb{R} \to Y$ by the equation $\boxed{f(t) = ty_0}$. Compute f'.
Define $g: \mathbb{R} \to Y$ by the equation $\boxed{g(t) = (\sin t)\, y_0}$.
Compute g'.

Sol:- (i) Here $f(t) = ty_0$

$$f(t+h) - f(t) = (t+h)\, y_0 - ty_0; \quad t \in \mathbb{R},\ h \in \mathbb{R},\ y_0 \in Y.$$

$$= \cancel{ty_0} + hy_0 - \cancel{ty_0}$$

$$= hy_0$$

$$\Rightarrow \quad f(t+h) - f(t) = hy_0$$

Define $A: \mathbb{R} \to Y$ s.t. $\boxed{Ah = hy_0}$

(a) To show A is linear.

$$A(ah_1 + bh_2) = (ah_1 + bh_2)\, y_0 \quad ; \quad h_1, h_2 \in \mathbb{R}$$
and $a, b \in \mathbb{R}$

$$= ah_1 y_0 + bh_2 y_0$$

$$= a\,Ah_1 + b\,Ah_2$$

$$\Rightarrow \boxed{A(ah_1 + bh_2) = a\,Ah_1 + b\,Ah_2} \rightarrow A \text{ is linear.}$$

(b) To show A is bounded. i.e; $\|A\| = M < \infty$

$$\|Ah\| = \|h y_0\|$$
$$= |h| \, \|y_0\|$$

Take $0 \neq h \in \mathbb{R}$

$$\frac{\|Ah\|}{|h|} = \|y_0\|$$

$$\Rightarrow \sup_{h \neq 0} \frac{\|Ah\|}{|h|} = \|y_0\|$$

$$\Rightarrow \|A\| = \|y_0\| \qquad \left(\because \sup \frac{\|Ax\|}{|x|} = \|A\| \right)$$

here y_0 be a fixed point of normed linear space Y. So its norm is finite.

$$\Rightarrow \|A\| = \|y_0\| < \infty$$

$\Rightarrow$ A is bounded map.

Claim: $\boxed{\lim_{h \to 0} \frac{\|f(t+h) - f(t) - Ah\|_Y}{\|h\|_{\mathbb{R}}} = 0}$

$$\Rightarrow \lim_{h \to 0} \frac{\|f(t+h) - f(t) - Ah\|_Y}{|h|}$$

$$= \lim_{h \to 0} \frac{\|h y_0 - h y_0\|_Y}{|h|} = \lim_{h \to 0} \frac{\|0\|_Y}{|h|} = 0$$

Now $\boxed{A = f'(t) = y_0} \to$ fixed

$A' : \mathbb{R} \to L(\mathbb{R}, Y)$

Let $f(t) = 2$	$f(t) = 2t$
$\boxed{f \equiv 2}$ $\downarrow$ fixed point	$\boxed{f \equiv 2I}$ $\downarrow$ not fixed point

$$\Rightarrow \boxed{f'(t) = f' = A = y_0} \quad \forall\, t \in \mathbb{R}$$

$$\Rightarrow \quad \underline{f' = y_0} \quad \forall\, t \in \mathbb{R}$$

(ii) Here $g(t) = (\sin t)\, y_0$

$$\boxed{g'(t) = (\cos t)\, y_0 = A}$$

Claim: $\displaystyle \lim_{h \to 0} \frac{\|g(t+h) - g(t) - Ah\|_Y}{\|h\|_{\mathbb{R}}} = 0$

$$\lim_{h \to 0} \frac{\|g(t+h) - g(t) - Ah\|_Y}{|h|}$$

$$= \lim_{h \to 0} \frac{\|(\sin(t+h))y_0 - (\sin t)y_0 - (\cos t)h y_0\|_Y}{|h|}$$

$$= \lim_{h\to 0} \frac{\|(\sin(t+h) - \sin t - (\cos t)h)y_0\|}{|h|}$$

$$= \lim_{h\to 0} \frac{|\sin(t+h) - \sin t - (\cos t)h| \, \|y_0\|}{|h|}$$

($\because$ If $V(F)$ is vector space & $\forall a \in F$, $\forall \alpha \in V$ then $\|a\alpha\| = |a| \, \|\alpha\|$)

$$= \lim_{h\to 0} \left| \frac{\sin(t+h) - \sin t - (\cos t)h}{h} \right| \|y_0\|$$

$$= \left| \lim_{h\to 0} \left(\frac{\sin(t+h) - \sin t - (\cos t)h}{h} \right) \right| \|y_0\|$$

($\because$ modulus fun is continuous)

$$= \left| \lim_{h\to 0} \left(\frac{\sin(t+h) - \sin t}{h} \right) - (\cos t) \right| \|y_0\|$$

$$= |\cos t - \cos t| \, \|y_0\| = |0| \, \|y_0\| = 0$$

($\because$ This is nothing but the derivative of $\sin t$ in one variable, which we know that $\cos t$.)

Define $A : \mathbb{R} \to Y$ s.t. $\boxed{Ah = (\cos t)\, h\, y_0}$

(a) To show A is linear

$$A(ah_1 + bh_2) = (\cos t)(ah_1 + bh_2)\, y_0 \quad ; a, b \in \mathbb{R},\ h_1, h_2 \in \mathbb{R}$$

$$= ((\cos t)ah_1 + (\cos t)bh_2)\, y_0$$

$$= a\,(\cos t)h_1 y_0 + b\,(\cos t)\,h_2 y_0$$

$$= a A h_1 + b A h_2$$

$$\Rightarrow \boxed{A(ah_1 + bh_2) = aAh_1 + bAh_2} \rightarrow \text{linear}$$

(b) To show A is bounded.

$$\|Ah\| = \|(\cos t)h\, y_0\|$$

$$= \|((\cos t)h)\, y_0\|$$

$$= |(\cos t)\,h|\;\|y_0\|$$

$$= |\cos t|\;|h|\;\|y_0\|$$

Take $0 \neq h \in \mathbb{R}$

$$\frac{\|Ah\|}{|h|} = |\cos t|\;\|y_0\|$$

$$\Rightarrow \sup_{h \neq 0} \frac{\|Ah\|}{|h|} = |\cos t|\;\|y_0\|$$

$$\Rightarrow \|A\| = |\cos t|\;\|y_0\|$$

here y_0 is a fixed point of normed linear space Y. So its norm is finite. And $|\text{const}|$ is also finite.

$$\Rightarrow \|A\| = |\text{const}| \, \|y_0\| < \infty$$

~~Define~~ $g' : \mathbb{R} \to L(\mathbb{R}, Y)$ defines as

$$g' = g'(t) = A \quad \forall\, t \in \mathbb{R}$$

where $A : \mathbb{R} \to Y$ as

$$A = (\text{const})\, y_0$$

$$\Rightarrow \boxed{g' = g'(t) = A = (\text{const})\, y_0} \quad \forall\, t \in \mathbb{R}$$

$\boxed{\text{P-17}}$ Define $f : C[0,1] \to C[0,1]$ by the equation $\boxed{[f(x)](t) = x(t) + \int_0^1 [x(s,t)]^2 \, ds}$.

Compute $f'(x)$.

Sol] Here $[f(x)](t) = x(t) + \int_0^1 [x(s,t)]^2 \, ds$

$$[f(x+h)](t) - [f(x)](t) = (x+h)(t) + \int_0^1 [(x+h)(s,t)]^2 ds - x(t) - \int_0^1 [x(s,t)]^2 ds$$

$$= x(t) + h(t) + \int_0^1 [x(st) + h(st)]^2\, ds$$

$$- x(t) - \int_0^1 [x(st)]^2\, ds$$

$$= \cancel{x(t)} + h(t) + \int_0^1 \left[(x(st))^2 + (h(st))^2 + 2\, x(st)\, h(st) \right] ds - \int_0^1 [x(st)]^2\, ds - \cancel{x(t)}$$

$$= h(t) + \cancel{\int_0^1 [x(st)]^2\, ds} + \int_0^1 [h(st)]^2\, ds + 2\int_0^1 x(st)\, h(st)\, ds - \cancel{\int_0^1 [x(st)]^2\, ds}$$

$$= \underbrace{h(t) + 2\int_0^1 x(st)\, h(st)\, ds} + \int_0^1 [h(st)]^2\, ds$$

Define $A : C[0,1] \to C[0,1]$

s.t. $\boxed{(Ah)(t) = h(t) + 2\int_0^1 x(st)\, h(st)\, ds}$

($\because$ Derivative is nothing but a linear map)

Claim: $$\lim_{h\to 0} \frac{\| f(x+h) - f(x) - Ah\|}{\|h\|} = 0$$

$$\| f(x+h) - f(x) - Ah\| = \sup_{t\in[0,1]} \left|\left(f(x+h) - f(x) - Ah\right)(t)\right|$$

$$= \sup_{t\in[0,1]} \left| (f(x+h))(t) - (f(x))(t) - (Ah)(t)\right|$$

$$= \sup_{t\in[0,1]} \left| \cancel{h(t)} + \cancel{2\int_0^1 x(st)\, h(st)\, ds} + \int_0^1 [h(st)]^2 ds - \cancel{h(t)} - \cancel{2\int_0^1 x(st)\, h(st)\, ds} \right|$$

$$= \sup_{t\in[0,1]} \left| \int_0^1 [h(st)]^2 ds \right|$$

$$\leq \sup_{t\in[0,1]} \int_0^1 \left| [h(st)]^2 \right| ds$$

$$= \sup_{t\in[0,1]} \int_0^1 \left| h(st) \cdot h(st) \right| ds$$

$$= \sup_{t\in[0,1]} \int_0^1 |h(st)|\, |h(st)|\, ds$$

$$\leq \int_0^1 \sup_{t\in[0,1]} |h(st)| \sup_{t\in[0,1]} |h(st)|\, ds$$

$$= \int_0^1 \|h\|\, \|h\|\, ds$$

$$= \|h\|^2 \int_0^1 ds$$

$$= \|h\|^2 (1)$$

$$= \|h\|^2$$

$$\Rightarrow \|f(x+h) - f(x) - Ah\| = \|h\|^2$$

$$\Rightarrow \lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = \lim_{h\to 0} \frac{\|h\|^{\cancel{2}}}{\cancel{\|h\|}}$$

$$= \lim_{h \to 0} \|h\| = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0$$

~~Define~~ $\boxed{(Ah)(t) = h(t) + 2\int_0^1 x(st)\,h(st)\,ds}$

(a) T.S. A is linear

$$(A(h_1+h_2))(t) = (h_1+h_2)(t) + 2\int_0^1 x(st)(h_1+h_2)(st)\,ds$$

$$= h_1(t) + h_2(t) + 2\int_0^1 x(st)\left(h_1(st) + h_2(st)\right)ds$$

$$= h_1(t) + h_2(t) + 2\int_0^1 x(st)\,h_1(st)\,ds + 2\int_0^1 x(st)\,h_2(st)\,ds$$

$$= h_1(t) + 2\int_0^1 x(st)\,h_1(st)\,ds + h_2(t) + 2\int_0^1 x(st)\,h_2(st)\,ds$$

$$= (Ah_1)(t) + (Ah_2)(t)$$

$$\Rightarrow \boxed{(A(h_1+h_2))(t) = (Ah_1)(t) + (Ah_2)(t)}$$

(b) To show A is bounded

$$\|Ah\| = \sup_{t\in[0,1]} |(Ah)(t)|$$

$$= \sup_{t\in[0,1]} \left(\left| h(t) + 2\int_0^1 x(st)\, h(st)\, ds \right| \right)$$

$$\leq \sup_{t\in[0,1]} \left(|h(t)| + \left| 2\int_0^1 x(st)\, h(st)\, ds \right| \right)$$

$$\left(\because |a+b| \leq |a| + |b| \right)$$

$$\leq \sup_{t\in[0,1]} \left(|h(t)| + 2\int_0^1 |x(st)\, h(st)|\, ds \right)$$

$$\leq \sup_{t\in[0,1]} |h(t)| + 2\int_0^1 \sup_{t\in[0,1]} |x(st)\, h(st)|\, ds$$

$$= \sup_{t\in[0,1]} |h(t)| + 2\int_0^1 \sup_{t\in[0,1]} |x(st)|\, |h(st)|\, ds$$

$$\leq \sup_{t\in[0,1]} |h(t)| + 2\int_0^1 \left(\sup_{t\in[0,1]} |x(st)| \sup_{t\in[0,1]} |h(st)| \right) ds$$

$$= \|h\| + 2\int_0^1 \|x\|\,\|h\|\,ds$$

$$\left(\because \sup_{t\in[0,1]} \|x\| = \|x\| \right)$$

$$= \|h\| + \|x\|\,\|h\|\,(2)\int_0^1 ds$$

$$= \|h\| + 2\|x\|\,\|h\|\,(1)$$

$$= \|h\|(1 + 2\|x\|)$$

$$\Rightarrow \|Ah\| \le \|h\|(1 + 2\|x\|)$$

Take $0 \neq h \in C[0,1]$

$$\Rightarrow \sup_{h\neq 0} \frac{\|Ah\|}{\|h\|} = 1 + 2\|x\| \quad \left(\because \|A\| = \sup_{h\neq 0} \frac{\|Ah\|}{\|h\|} \right)$$

$$\Rightarrow \|A\| \le 1 + 2\|x\|$$

$\Rightarrow$ A is bounded.