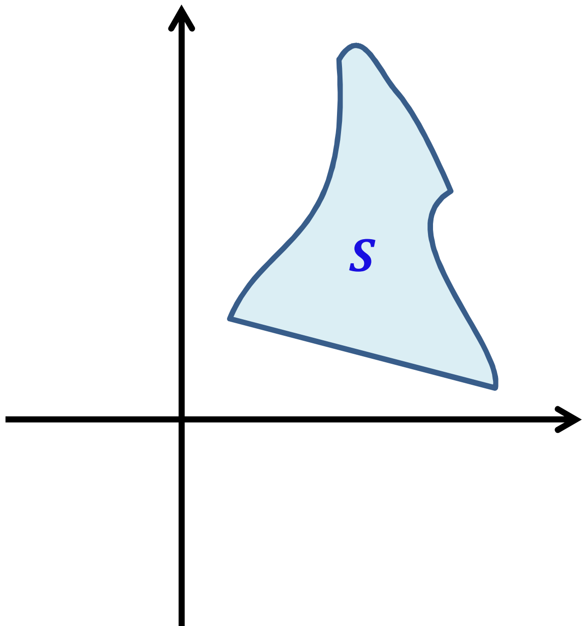
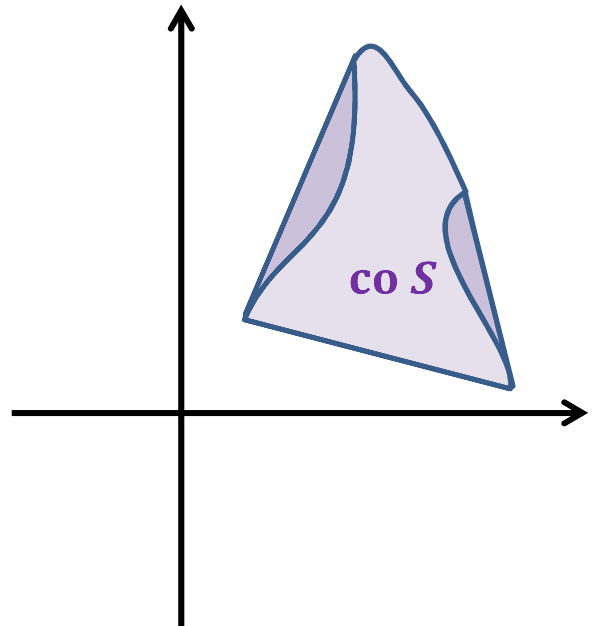
R-07
Convex and Nonsmooth Analysis

Convex Hull of a Set

Convex hull of a set $S \subseteq \mathbb{R}^n$, denoted by co$S$, is the intersection of all convex sets containing S.





$$\text{co}S := \cap \{C : C \text{ is a convex set}, S \subseteq C\}.$$

Convex Hull in terms of Convex Combinations

Theorem If S is a set in $\mathbb{R}^n$ then

$$\text{co}S = \left\{x \in \mathbb{R}^n : \text{for some } k \in \mathbb{N}, \exists\, x_1, \dots, x_k \in S, \alpha \in \Delta_k \text{ such that } \sum_{i=1}^k \alpha_i x_i = x\right\}$$

Proof Let T denote the set on the right hand side. Let $x \in T$.

Let C be a convex set containing S. Since C is a convex set it contains all convex combinations of elements of S. Hence, $x \in C$.

Thus, $T \subseteq C$ which implies that

$$T \subseteq \cap \{C : C \text{ is a convex set}, S \subseteq C\} = \text{co}S.$$

Conversely, we see that $S \subseteq T$, as $x = 1.x \in T$. This implies that co$S \subseteq$ co$T$.

It is enough to show that T is a convex set. Let $x, y \in T$ and $\lambda \in [0,1]$.

As $x \in T$, thereofore there exist $k \in \mathbb{N}, x_1, \dots, x_k \in S, \alpha \in \Delta_k$ such that $\sum_{i=1}^k \alpha_i x_i = x$.Similarly, as $y \in T$ thereofore for some $m \in \mathbb{N}$, there exist $y_1, \dots, y_m \in S, \beta \in \Delta_m$ such that $\sum_{j=1}^m \beta_j y_j = y$. Now for $\lambda \in [0,1]$ we have

$$\begin{aligned}(1-\lambda)x + \lambda y &= (1-\lambda)\sum_{i=1}^k \alpha_i x_i + \lambda \sum_{j=1}^m \beta_j y_j \\ &= \sum_{i=1}^k (1-\lambda)\alpha_i x_i + \sum_{j=1}^m \lambda \beta_j y_j\end{aligned}$$

continued

Clearly, $(1-\lambda)\alpha_i \geq 0, i = 1,2, \dots, k, \lambda\beta_j \geq 0, j = 1,2, \dots, m$ and

$$\sum_{i=1}^{k}(1-\lambda)\alpha_i + \sum_{j=1}^{m}\lambda\beta_j = (1-\lambda)\sum_{i=1}^{k}\alpha_i + \lambda\sum_{j=1}^{m}\beta_j = (1-\lambda) + \lambda = 1.$$

Hence, $(1-\lambda)x + \lambda y \in T$ for $\lambda \in [0,1]$.

Theorem If S is a set in $\mathbb{R}^n$ then $\text{co}S$ is the smallest convex set containing S.

Proof Let T be convex set containing S. Clearly,

$$\text{co}S = \cap \{C: C \text{ is a convex set}, S \subseteq C\} \subseteq T.$$

Affine Combination

Let $\{x_i\}_{i=1}^{k}$ be a finite set of points in $\mathbb{R}^n$. An affine combination of these points is any point of the form

$$x = \sum_{i=1}^{k} \lambda_i x_i, \; \sum_{i=1}^{k} \lambda_i = 1.$$

Set of all affine combinations of two points is the line passing through those points. Set of all affine combinations of three noncollinear points is the plane passing through those points.

Lemma A set $A \subseteq \mathbb{R}^n$ is an affine manifold if and only if it contains all affine combinations of points in A.

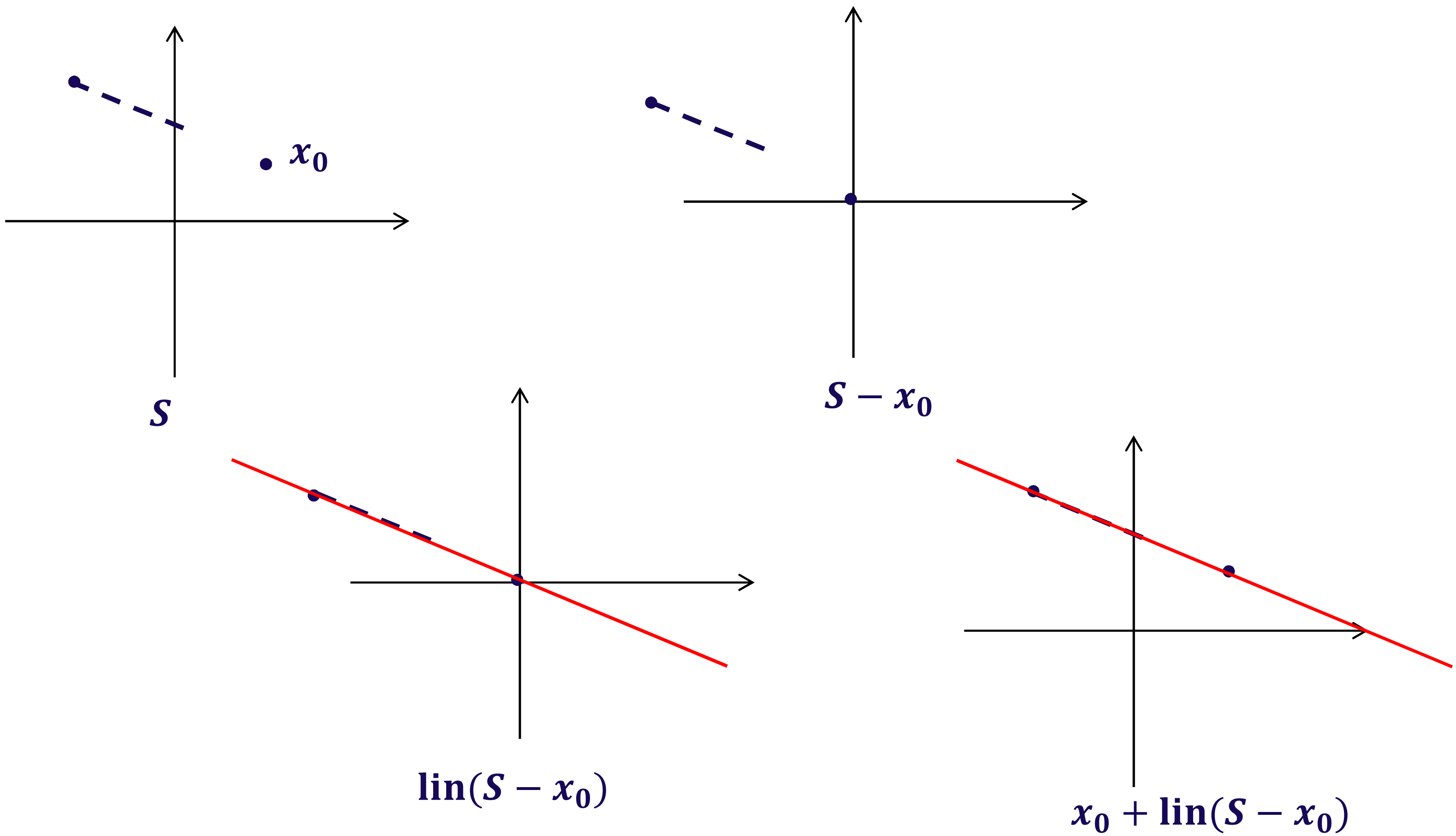
Affine hull of a set $S \subseteq \mathbb{R}^n$, denoted by aff$S$, is the intersection of all affine sets containing S.

Theorem Affine hull of a set S in $\mathbb{R}^n$ is the smallest affine set containing S.

Convex Hull and Linear Hull

For any $x_0 \in S$ we have

$$\mathrm{aff}S = x_0 + \mathrm{lin}(S - x_0).$$



Affinely Independent Points

A set of $k+1$ points $x_0, x_1, x_2, \dots, x_k$ are said to be affinely independent if the set

$$x_0 + \mathrm{lin}\{x_1 - x_0, x_2 - x_0, \dots, x_k - x_0\}$$

has full dimension k. This is equivalent to the fact that

$$\mathrm{lin}\{x_1 - x_0, x_2 - x_0, \dots, x_k - x_0\}$$

has full dimension k, that is, $x_1 - x_0, x_2 - x_0, \dots, x_k - x_0$ are k linearly independent vectors. So

$$\sum_{i=1}^{k} \alpha_i (x_i - x_0) = 0 \implies \alpha_i = 0, i = 1,2,\dots,k.$$

$$\Updownarrow$$

$$\sum_{i=1}^{k} \alpha_i x_i - \left(\sum_{i=1}^{k} \alpha_i\right) x_0 = 0 \implies \alpha_i = 0, i = 1,2,\dots,k.$$

$$\Updownarrow$$

$$\sum_{i=1}^{k} \alpha_i x_i + \alpha_0 x_0 = 0,\ \alpha_0 = -\sum_{i=1}^{k} \alpha_i \implies \alpha_i = 0, i = 1,2,\dots,k.$$

$$\Updownarrow$$

$$\sum_{i=0}^{k} \alpha_i x_i = 0, \sum_{i=0}^{k} \alpha_i = 0 \implies \alpha_i = 0, i = 0,1,2,\dots,k.$$

A set of $k+1$ points $x_0, x_1, x_2, \dots, x_k$ are affinely independent if

$$\sum_{i=0}^{k} \alpha_i x_i = 0, \sum_{i=0}^{k} \alpha_i = 0 \implies \alpha_i = 0, i = 0,1,2,\dots,k.$$

Example

- Are the set of 4 points $\{\mathbf{0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ in $\mathbb{R}^3$, where

$$\mathbf{0} = (0,0,0), \boldsymbol{e_1} = (1,0,0), \boldsymbol{e_2} = (0,1,0), \boldsymbol{e_3} = (0,0,1),$$

affinely independent?

Let

$$\alpha_0\mathbf{0} + \sum_{i=1}^{3} \alpha_i \boldsymbol{e_i} = \mathbf{0}, \quad \alpha_0 + \sum_{i=1}^{3} \alpha_i = 0.$$

$$\alpha_0\mathbf{0} + \sum_{i=1}^{3} \alpha_i \boldsymbol{e_i} = \mathbf{0} \Longrightarrow (\alpha_1, \alpha_2, \alpha_3) = \mathbf{0} \Longrightarrow \alpha_i = 0, i = 1,2,3.$$

As $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_0 + \sum_{i=1}^{3} \alpha_i = 0$ we have $\alpha_0 = 0$.

- Are the set of 4 points $\{\boldsymbol{a}, \boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ in $\mathbb{R}^3$, where $\boldsymbol{a} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, affinely independent?

$$\alpha_0\boldsymbol{a} + \sum_{i=1}^{3} \alpha_i \boldsymbol{e_i} = \mathbf{0}, \quad \alpha_0 + \sum_{i=1}^{3} \alpha_i = 0$$

holds for $\alpha = (3, -1, -1, -1)$.

Example

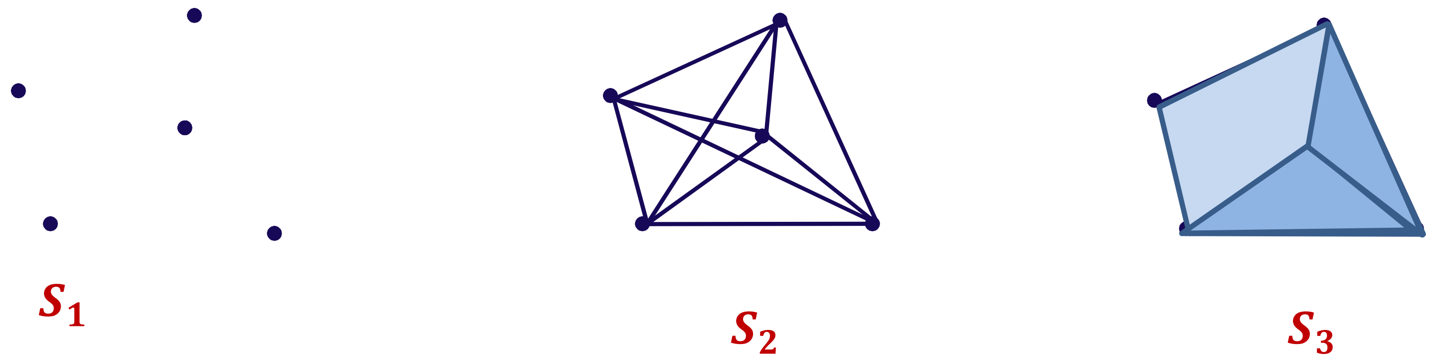
Let $S = \{x_1, x_2, \dots, x_m\} \subseteq \mathbb{R}^n$. Then

$$\text{co}S = \{x \in \mathbb{R}^n \colon \alpha \in \Delta_m, x = \sum_{i=1}^{m} \alpha_i x_i\}\ .$$

Do we have to take $\alpha \in \Delta_m$ always? If $m \leq n$, then it is fine. What if $m > n$?

For example consider $S = \{x_1, x_2, \dots, x_5\} \subseteq \mathbb{R}^2$.

Let S_i be the set of all convex combination of i points of S. Then $S = S_1$.



Hence we only need to consider $\alpha \in \Delta_3$ to write convex hull of a finite set of points in $\mathbb{R}^2$.

What if we have a set in $\mathbb{R}^3$?

What if S is not finite?

Carathéodory Theorem

Theorem Any $x \in \text{co}S \subseteq \mathbb{R}^n$ can be expressed as a convex combination of $n+1$ elements of S.

Proof Let $x = \sum_{i=1}^{k} \alpha_i x_i, \alpha \in \Delta_k,\ x_i \in S, \alpha_i > 0, i = 1,2,..k$. If $k \leq n+1$ we are done. Let $k > n+1$. We will show that x can be expressed as a convex combination of $k-1$ elements of $\{x_1, x_2, \dots, x_k\}$.

Since $k > n+1$, $x_1, x_2, \dots, x_k$ are affinely dependent. We can find $\delta_1, \delta_2, \dots, \delta_k$ not all zero such that

$$\sum_{i=1}^{k} \delta_i x_i = 0, \sum_{i=1}^{k} \delta_i = 0.$$

Hence,

$$x = \sum_{i=1}^{k} (\alpha_i - t\delta_i) x_i, \text{ for } t \geq 0. \qquad (1)$$

Let $t^* = \max\{t \geq 0, \alpha_i - t\delta_i \geq 0, \delta_i > 0, i \in \{1,2,\dots,k\}\}$

$$= \min_{\delta_j > 0} \frac{\alpha_j}{\delta_j} = \frac{\alpha_r}{\delta_r}, \text{ for some } r \in \{1,2,\dots,k\}.$$

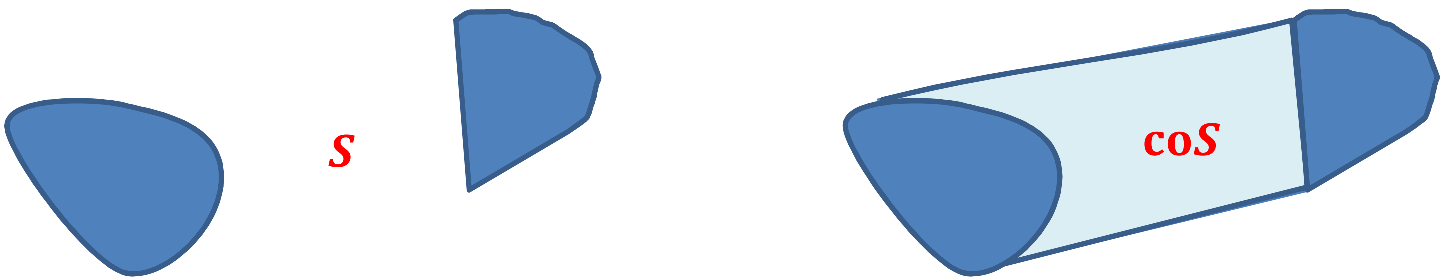
Set $\alpha_i^* = \alpha_i - t^*\delta_i$. Clearly, $\alpha_i^* \geq 0$ with $\alpha_r^* = 0$. Now

$$\sum_{\substack{i=1 \\ i \neq r}}^{k} \alpha_i^* = \sum_{i=1}^{k} \alpha_i^* = \sum_{i=1}^{k} (\alpha_i - t^*\delta_i) = \sum_{i=1}^{k} \alpha_i - t^* \sum_{i=1}^{k} \delta_i = 1 - 0 = 1.$$

Taking $t = t^*$ in (1) we get $x = \sum_{\substack{i=1 \\ i \neq r}}^{k} \alpha_i^* x_i$. If $k-1 = n+1$ the proof is finished. Otherwise continue the process of expressing x as a convex combination of $k-2, k-3, \dots$ elements until one of them equals $n+1$.

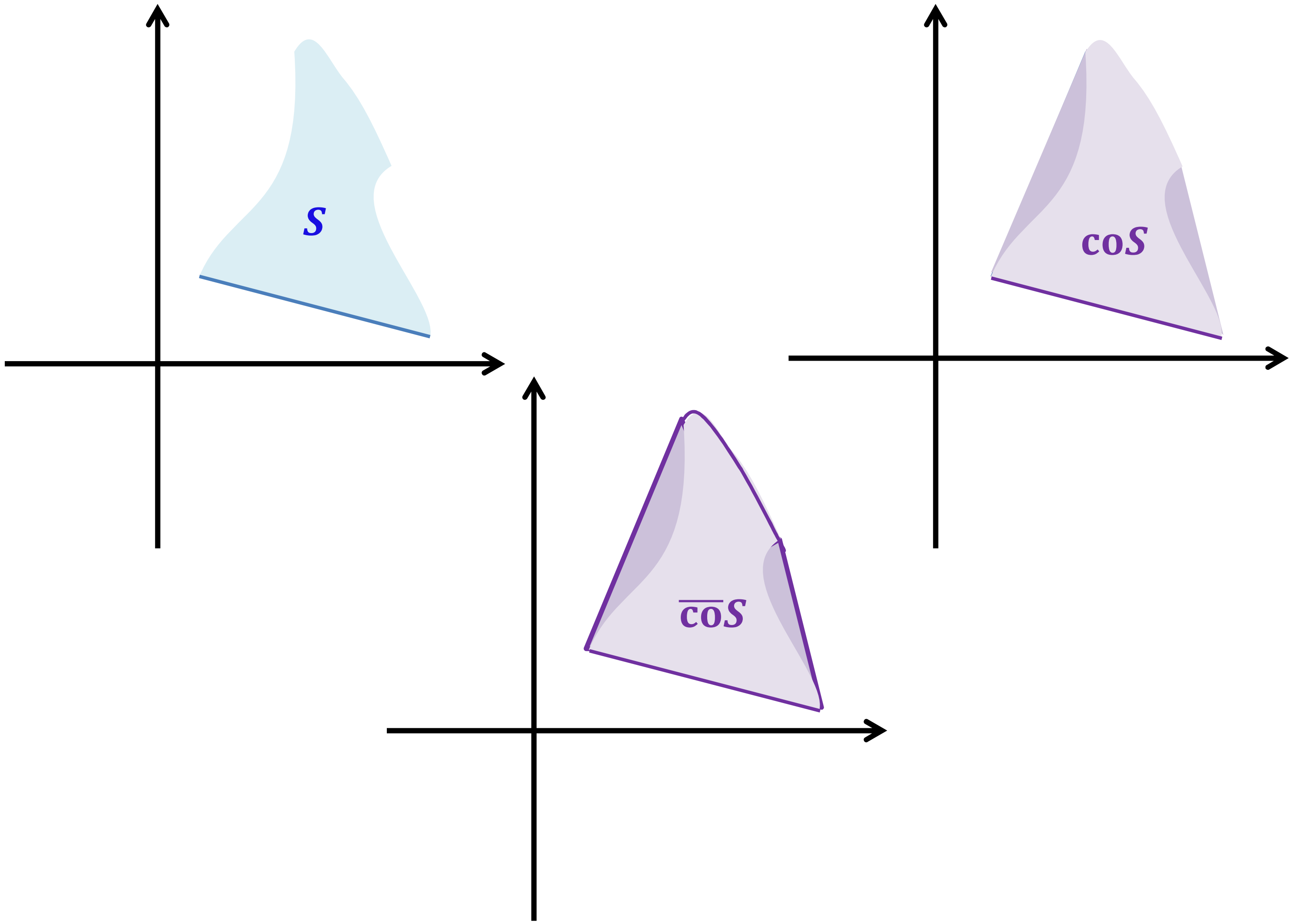
What if there are connected componenets?

Theorem If $S \subseteq \mathbb{R}^n$ has no more than n connected components, then any $x \in \mathrm{co}S$ can be expressed as a convex combination of n elements of S.



Closed Convex Hull of a Set

Closed convex hull of a set $S \subseteq \mathbb{R}^n$, denoted by $\overline{\mathrm{co}}S$, is the intersection of all closed convex sets containing S.



Closed Convex Hull

Theorem The closed convex hull $\overline{\mathrm{co}}S$ is the closure of the convex hull of S, that is,

$$\overline{\mathrm{co}}S = \mathrm{cl}(\mathrm{co}S).$$

Proof The set $\mathrm{cl}(\mathrm{co}S)$ is a closed convex set containing S. Hence, $\overline{\mathrm{co}}S \subseteq \mathrm{cl}(\mathrm{co}S)$ as $\overline{\mathrm{co}}S$ is the intersection of all closed convex set containing S.

Conversely, let C be a closed convex set containing S. Since C is convex we have co$S \subseteq C$. As C is closed we have

$$\mathrm{cl}(\mathrm{co}S) \subseteq C.$$

Hence,

$$\mathrm{cl}(\mathrm{co}S) \subseteq \cap \{C: C \text{ is a closed convex set}, S \subseteq C\} = \overline{\mathrm{co}}S.$$

Remark $\overline{\mathrm{co}}$ does not distinguish S from co$S$ and cl$S$, that is,

$$\overline{\mathrm{co}}S = \overline{\mathrm{co}}(\mathrm{co}S) = \overline{\mathrm{co}}(\mathrm{cl}S).$$