Fixed-Point Theorems-

Defn. A topological space X has the fixed point property if every continuous map $f: X \to X$ has a fixed point.

Thm. 1 - Brower's fixed-point theorem - Every compact convex set in $\mathbb{R}^n$ has the fixed-point property.

Thm. 2 - If a topological space has the fixed point property, then the same is true for every space homeomorphic to it.

Proof. Let X and Y be homeomorphic spaces.

$\Rightarrow \exists$ a homeomorphism $h: X \to Y$ (i.e. a cont. bijection with cont. inverse)

Suppose X has fixed point property.

To prove - Y has the fixed point property.

Let $f: Y \to Y$ be a continuous map.

Then the map $h^{-1} \circ f \circ h$ is continuous from X to X.

(Composition of cont. maps)

$$h^{-1} \circ f \circ h : X \xrightarrow{h} Y \xrightarrow{f} Y \xrightarrow{h^{-1}} X$$

$\Rightarrow h^{-1} \circ f \circ h$ has a fixed point, say x. ($\because X$ has fixed point property)

$\Rightarrow \quad h^{-1} \circ f \circ h(x) = x$

$\Rightarrow \quad h(h^{-1} \circ f \circ h(x)) = h(x) \quad \Rightarrow \quad f \circ h(x) = h(x) \qquad (\because h \circ h^{-1} = I_Y)$

$\Rightarrow h(x)$ is a fixed point of f.

$\Rightarrow Y$ has the fixed point property.

Locally Convex linear topological Hausdorff Space -

- Topological Space - $X \neq \phi$, (X, τ) with four axioms.
- Hausdorff Space - A top. space is said to be hausdorff if for every $x, y \in X$ $x \neq y$, $\exists$ two open sets $U, V \subseteq X$ s.t. $x \in U$ & $y \in V$ (disjoint) and $U \cap V = \phi$.

  eg. - $(\mathbb{R}, \tau_u)$

- Linear Top. Space -

  A linear topological space is a pair (X, τ) in which X is a linear space (i.e. vector space, normed space) and τ is a topology on X such that algebraic operations in X are continuous

  i.e. the two maps
  $X \times X \to X$
  $(x, y) \mapsto x + y$ (vector addition)
  $\mathbb{R} \times X \to X$
  $(\lambda, x) \mapsto \lambda x$ (scaler multiplication)

  are continuous.

- Thm. - (A) A linear topological space is a Hausdorff space if and only if 0 is the only element common to all neighbourhoods of 0.

- Th. (B) - - In a lin. top. space, V be a nbd. of 0 then $\exists$ a sym. nbd. U of 0 s.t. $U + U + U \subseteq V$.

- Locally Convex linear topological Space --

  If the nbd.s of 0 in a linear topological space have a base consisting of convex sets, then the space is said to be locally convex
  (i.e. 0 has a local base consisting of convex sets)

  eg. - (i) Every normed space is locally convex.

  (open balls centred at 0, with diff. radius form the local base of convex sets.)

  (ii) $(\mathbb{R}, \tau_u)$, $(-\delta, \delta)$ around '0'

Lemma - If K is a Compact set in a locally Convex linear topological space, and if U is Symmetric, Convex, open nbd. of 0, then there is a finite set F in K and a Continuous map P from K to the Convex hull of F such that $x - Px \in U \ \forall x \in K$.

i.e. $K \subseteq X$ - locally convex linear top. space

K ↓ Compact

$0 \in U \subseteq X$

↳ Sym., Convex, open nbd.

Then $\exists$ finite $F \subseteq K$ and $P : K \longrightarrow Co(F) \subseteq K$

Cont.^{n}

s.t. $x - Px \in U$

Nets. (generalised Seq.^{n}) - A net is a fun. on a directed set.

- A partially ordered set is a pair $(\mathcal{D}, <)$ in which $\mathcal{D}$ is a set and $<$ is a relation obeying these axiom :

a. $a < a$

b. If $a < b$ and $b < c$ then $a < c$.

A directed Set is a Partially ordered set in which an additional axiom is required :

c. Given a and b in $\mathcal{D}$, there is $c \in \mathcal{D}$ st. $a < c$ and $b < c$

for eg. $(\mathbb{N}, \leq)$ is directed set.

- A net (x_α) is eventually in a set V if there is a β Such that $x_\alpha \in V$ whenever $\beta < \alpha$.

- If a net is eventually in every nbd. of a point y, then we say that then, we say that the net Converges to y.

Thm. (3). The Schauder - Tychonoff fixed-point Theorem -

Every Compact Convex set in a locally Convex linear topological Hausdorff Space has the fixed-point property.

proof. Let K be such a set.

i.e. K is Compact and Convex Subset of a locally Convex haus. T.V.S

and let $f: K \to K$ be a Continuous map

we denote the family of all Convex, symmetric, open nbds. of 0 by $\{U_\alpha : \alpha \in A\}$.

(This family is nonempty, ∵ here our space is locally Convex.

$\Rightarrow$ 0 has a Convex local base

$\Rightarrow$ There are Convex, open nbd. of 0.

for sym. $0 \in \underbrace{U \cap (-U)}_{\text{sym}} \subseteq U$)

The set A is simply an index set, which we partially order by writing $\alpha \geq \beta$ when $U_\alpha \subset U_\beta$. (ie. $(\;(U_\alpha)\;U_\beta)$ then $\alpha > \beta$)

Thus, ordered A becomes a directed set. (for every α, β ∃ γ s.t. $\gamma \geq \alpha$ & $\gamma \geq \beta$)

(Suitable as the domain of a net).

Since K is Compact $\Rightarrow$ the map f is uniformly Continuous,

therefore, Corresponds to any $\alpha \in A$ ∃ an $\alpha' \in A$ s.t. $U_{\alpha'} \subset U_\alpha$

and $f(x) - f(y) \in U_\alpha$ whenever $x - y \in U_{\alpha'}$ ——— (i)

(∵ If f is u.c. btw. linear top. spaces, by the defn of u.c. btw. linear T.S., we have corresponding to any nbd. V of in K ∃ a nbd. $U_{\alpha'}$ of 0 in K s.t. (i) holds) for any $\alpha \in A$, The above lemma provides a Continuous map P_α Such that, $P_\alpha(K)$ is a Compact, Convex, finite dim. subset of K.

[ie. for any $\alpha \in A$, $\exists$ U_α

$$P_\alpha : K \to \underbrace{Co(F)}_{\text{finite set in } K} = P_\alpha(K) \quad \text{s.t.} \quad x - P_\alpha x \in U_\alpha$$

$P_\alpha(K)$ is compact ($\because$ Cont. Image of compact set)

$P_\alpha(K) = Co(F)$ — (Convex hull of finit points)
is convex

$P_\alpha(K) = Co(F)$ is finite dim. ($\because$ Convex hull of finite points)]

This map has the further property that $x - P_\alpha x \in U_\alpha$ for each $x \in K$.

The Composition $P_\alpha \circ f$ maps $P_\alpha(K)$ into itself.

[$\because$ $P_\alpha \circ f : K \xrightarrow{f} K \xrightarrow{P_\alpha} P_\alpha(K)$

$P_\alpha \circ f \big|_{P_\alpha(K)} : P_\alpha(K) \xrightarrow{f} K \xrightarrow{P_\alpha} P_\alpha(K)$]

Hence, by the Brouwer Fixed-point theorem,

$P_\alpha \circ f$ has a fixed point. say z_α in $P_\alpha(K)$.

[As $P_\alpha(K) = Co(F)$, which is a Convex hull of finite points. (WLOG n points)
which is a Compact, Convex subset of $\mathbb{R}^n$.
and $P_\alpha \circ f$ is cts. (composition of cts. maps)]

By Compactness of K, the net $(z_\alpha \mid \alpha \in A)$ has a cluster point z in K

($\because$ (z_α) is a net in K ie, $\phi : A \to K$ s.t. $\phi(\alpha) = z_\alpha$ (fixed pt of $P_\alpha \circ f$)
As K is Compact $\Rightarrow$ (z_α) has a cluster pt.)

Claim. z is a fixed point of f.

Consider,

$$f(z)-z = [f(z)-f(z_\alpha)] + [f(z_\alpha) - P_\alpha f(z_\alpha)] + [P_\alpha f(z_\alpha) - z]$$

$$= [f(z)-f(z_\alpha)] + [f(z_\alpha) - P_\alpha f(z_\alpha)] + [z_\alpha - z] \quad (\because P_\alpha f(z_\alpha) = z_\alpha)$$

(ii)

now, for any $\beta \in A$, we can select $\alpha \in A$ s.t. $\alpha \geq \beta$

($\because$ A is directed)

and $z - z_\alpha \in U_{\beta'}$ then $f(z) - f(z_\alpha) \in U_\beta$ (by (i))

($\because$ corr. to $\beta \in A$ $\exists$ $\beta' \in A$ s.t. $U_{\beta'} \subset U_\beta$

and $x - y \in U_{\beta'}$ whenever $f(x) - f(y) \in U_\beta$

here take $x = z$, $y = z_\alpha$)

Also, $f(z_\alpha) - P_\alpha f(z_\alpha) \in U_\alpha \subset U_\beta$. ($\because$ $\alpha \geq \beta \Rightarrow U_\alpha \subset U_\beta$)

[As $z_\alpha \in P_\alpha(K) \subseteq K$

$\Rightarrow$ $f(z_\alpha) \subseteq K$ and $x - P_\alpha x \in U_\alpha$ $\forall$ $x \in K$

$\Rightarrow$ $f(z_\alpha) - P_\alpha(f(z_\alpha)) \in U_\alpha$]

finally,

$z - z_\alpha \in U_{\beta'} \subset U_\beta$

by Eq.^n (2), we have, $f(z) - z \in 3U_\beta$.

Since, β was an arbitrary element of A

And for every nbd V of 0 $\exists$ a sym. nbd. U s.t. $U+U+U \subset V$. (by thm. (B))

And, here our space is linear top. hausdorff space.

So, by thm (A), 0 is the only common element to all nbd. of '0'.

and we have, $f(z) - z \in 3U_\beta$ for any $\beta \in A$

$\Rightarrow f(z) - z \in V$ for any arb. nbd. of '0'.

$\Rightarrow f(z) - z = 0$

$\Rightarrow f(z) = z$.

(Hence the Conclusion)

Corollary. If a continuous map is defined on a domain $\mathcal{D}$ in a locally convex linear topological hausdorff space and takes values in a compact, convex subset of $\mathcal{D}$, then it has a fixed point.

Proof Let $F : \mathcal{D} \to K$ be continuous map.

where $\mathcal{D} \subseteq X$ (X is loc. convex lin. top. hausd. space)

and K is compact convex subset of $\mathcal{D}$,

then $F|_K : K \to K$ is cts. map from K to K.

and K is compact convex subset of X

By Schauder - Tychonoff Thm., $F|_K$ has a fixed point

$\Rightarrow$ F has a fixed point.