2. Let f be a Frechet - differentiable function from a Hilbert space X into $\mathbb{R}$. The gradient of f at x is a vector $v \in X$ such that $f'(x)h = \langle h, v \rangle \quad \forall h \in X$.
Prove that such a v exists. (It depends on x)
Illustrate with $f(x) = \langle a, x \rangle^2$, $a \in X$ and fixed.

Solⁿ:- Existence of v

Given f is $X \longrightarrow \mathbb{R}$ is frechet - differentiable f^n
Then for each $x \in X$ $\exists$ a bounded linear map $A : X \longrightarrow \mathbb{R}$
such that
$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - Ah \|}{\|h\|} = 0$$
i.e $f'(x) = A_{f,x}$ (A depends on x and f)

Since $f'(x) = A : X \longrightarrow \mathbb{R}$ is a bounded linear functional, by "Riesz representation theorem"

[Every bounded linear functional f on a Hilbert Space H can be represented in terms of the inner product, namely
$$f(x) = \langle x, z \rangle \quad \forall x \in H$$
where z dependent on f is uniquely determined by f and $\|z\| = \|f\|$]

we have
$$f'(x)h = Ah = \langle h, v \rangle \quad \forall h \in X$$
where v depends on A (which in turn is dependent on x)

(ii) $f : X \longrightarrow \mathbb{R}$ defined as
$f(x) = \langle a, x \rangle^2$, where $a \in X$ is fixed

Claim 1: f is fréchet differentiable function.

For $x \in X$

$$f(x+h) - f(x) = \langle a, x+h\rangle^2 - \langle a, x\rangle^2$$
$$= (\langle a, x\rangle + \langle a, h\rangle)^2 - \langle a, x\rangle^2$$
$$= \langle a, x\rangle^2 + \langle a, h\rangle^2 + 2\langle a, x\rangle\langle a, h\rangle - \langle a, x\rangle^2$$
$$= \langle a, h\rangle^2 + 2\langle a, x\rangle\langle a, h\rangle$$

Let $Ah = 2\langle a, x\rangle\langle a, h\rangle$ then, (A is bounded by linear)

and $f(x+h) - f(x) - Ah = \langle a, h\rangle^2$

$$\frac{\| f(x+h) - f(x) - Ah\|}{\|h\|} = \frac{|\langle a, h\rangle^2|}{\|h\|} \le \frac{\|a\|^2\|h\|^2}{\|h\|} \quad \text{(By Cauchy Schwarz)}$$
$$= \|a\|^2\|h\| \longrightarrow 0 \quad \text{as } h \to 0.$$

$$\therefore \boxed{f'(x)h = 2\langle a, x\rangle\langle a, h\rangle = Ah}$$

Claim: A is bounded by linear

$Ah = 2\langle a, x\rangle\langle a, h\rangle = \langle 2a\langle a, x\rangle, h\rangle$ $\quad (\because \langle a, x\rangle \in \mathbb{R})$

(i) $A(\alpha h) = \langle 2\langle a, x\rangle a, \alpha h\rangle = \alpha\langle 2\langle a, x\rangle a, h\rangle$
$$= \alpha\, 2\langle a, x\rangle\langle a, h\rangle$$

(ii) $A(h_1+h_2) = \langle 2\langle a, x\rangle a, h_1+h_2\rangle$
$$= \langle 2\langle a, x\rangle a, h_1\rangle + \langle 2\langle a, x\rangle a, h_2\rangle$$
$$= 2\langle a, x\rangle\langle a, h_1\rangle + 2\langle a, x\rangle\langle a, h_2\rangle$$
$$= Ah_1 + Ah_2$$

Hence A is linear

Now, $|Ah| = |2\langle a, x\rangle\langle a, h\rangle| = 2|\langle a, x\rangle||\langle a, h\rangle|$
$$\le 2\|a\|\|x\|\|a\|\|h\| \quad \text{(By cauchy Schwarz)}$$
$$= 2\|a\|^2\|x\|\|h\|$$

$\Rightarrow$ $\|A\| \le 2\|a\|^2\|x\|$ $\quad$ Hence A is bounded.

~~Claim~~ To find: Gradient of f

For $x \in X$

$$f'(x)h = Ah = 2\langle a,x\rangle\langle a,h\rangle$$
$$= \langle 2\langle a,x\rangle a, h\rangle$$
$$= \langle h, 2\langle a,x\rangle a\rangle \quad (\because X \text{ is real Hilbert space})$$

Let $v = 2\langle a,x\rangle a$, $v \in X$, where $a \in X$ is fixed

then v depends on x and is the gradient of f at x.

4. Let X, Y and Z be normed linear spaces. Prove that if $f: X \longrightarrow Y$ is differentiable and if $A: Y \longrightarrow Z$ is a bounded linear map, then $(A \circ f)' = A \circ f'$. $A \circ f: X \longrightarrow Z$

Solⁿ: $f: X \longrightarrow Y$ is differentiable. So, for each $x \in X$ $\exists$ a bounded linear operator (say) $B: X \longrightarrow Y$ such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Bh\|_Y}{\|h\|_X} = 0$$

Claim: $A: Y \rightarrow Z$ is Fréchet differentiable.

Let $y \in Y$ then,

$$A(y+h) - A(y) = A(y+h-y) \quad (\because A \text{ is linear})$$
$$= Ah$$

$$\lim_{h\to 0} \frac{\|A(y+h) - A(y) - Ah\|}{\|h\|} = 0$$

Thus $\boxed{A'_y = A}$

i.e Frechet derivative of A at each point is A itself.

Now for $x \in X$

$$\frac{\|(A\circ f)(x+h) - (A\circ f)(x) - (A\circ B)h\|}{\|h\|} = \frac{\|A(f(x+h)) - A(f(x)) - A(Bh)\|}{\|h\|}$$

$$= \frac{\|A(f(x+h) - f(x) - Bh)\|}{\|h\|} \quad (\because A \text{ is linear})$$

$$\leq \frac{\|A\| \, \|f(x+h) - f(x) - Bh\|}{\|h\|}$$

Thaking limit $h \to 0$ on b/s we get

$$\lim_{h\to 0} \frac{\|(A\circ f)(x+h) - (A\circ f)(x) - (A\circ B)h\|}{\|h\|} = 0 \quad (\because f \text{ is frechet differentiable})$$

$\therefore$ $A\circ f$ is F-differentiable at x and $(A\circ f)'(x) = A\circ B$
$= A\circ f'(x)$

As $x \in X$ was arbitrary $\Rightarrow$ $A\circ f$ is frechet differentiable on whole X

and $\boxed{(A\circ f)' = A\circ f'}$