Question-21 Refer to the definition of the Frechet derivative. If the bdd linear map A satisfies the weaker condition.

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \| f(x+\lambda h) - f(x) - \lambda A h \|_Y = 0$$

for every $h \in X$, then f is said to be Gâteaux differentiable at x, and A is the Gâteaux derivative at x.

Prove that - If f is Frechet differentiable at x, then it is Gateaux differentiable at x, and the two derivatives are equal.

Proof:- Assume f is Frechet differentiable at x_0.

i.e.

[$f : D \to Y$ is a mapping from an open set $D \subseteq X$ where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear space.

Let $x_0 \in D$.

f is said to be Frechet differentiable iff $\exists$ a bdd linear map $A : X \to Y$ $(A \in B(X,Y))$ such that

$$\lim_{h \to 0} \frac{\| f(x_0 + h) - f(x_0) - Ah \|_Y}{\|h\|_X} = 0.$$

and A is called the frechet derivative of f at x_0.]

fix $0 \neq h \in X$.

Consider,

$$\frac{1}{|\lambda|} \| f(x_0 + \lambda h) - f(x_0) - \lambda A(h) \|_Y$$

$$= \frac{1}{|\lambda|} \| f(x_0 + \lambda h) - f(x_0) - A(\lambda h) \|_Y \quad (\because A \text{ is linear})$$

$$= \frac{\|h\|}{|\lambda| \, \|h\|} \| f(x_0 + \lambda h) - f(x_0) - A(\lambda h) \|_Y$$ [multiply by $\|h\|$ in numerator and denominator]

$$= \|h\| \left[\frac{1}{\|\lambda h\|_X} \| f(x_0 + \lambda h) - f(x_0) - A(\lambda h) \|_Y \right]$$

as f is Frechet differentiable at x_0 then

$$\frac{\| f(x_0 + \lambda h) - f(x_0) - A(\lambda h) \|_Y}{\|\lambda h\|_X} \to 0$$

as $\|\lambda h\|_X \to 0$.

$$\Rightarrow \|h\| \left[\frac{1}{\|\lambda h\|_X} \| f(x_0 + \lambda h) - f(x_0) - A(\lambda h) \|_Y \right]$$

$\to 0$ as $\|\lambda h\|_X \to 0$ iff $\lambda \to 0$.

(since h is fixed)

Hence, $\lim_{\lambda \to 0} \frac{1}{\lambda} \| f(x_0 + \lambda h) - f(x_0) - \lambda A(h) \|_Y = 0$

So, f is Gateaux differentiable at x_0 with Gateaux derivative A and hence both the derivative (Gateaux and Frechet) are equal.

Gateaux differentiable does not imply Frechet differentiable.

Eg:- Define $f: \mathbb{R}^2 \to \mathbb{R}$

as

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

f is Gateaux differentiable at $(0,0)$.

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \left(f((0,0)+\lambda(h,k)) - f(0,0) \right)$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left(f(\lambda h, \lambda k) - f(0,0) \right)$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\frac{\lambda^3 h^3}{\lambda^2(h^2+k^2)} - 0 \right]$$

$$= \lim_{\lambda \to 0} \frac{\lambda h^3}{\lambda (h^2+k^2)} = \frac{h^3}{h^2+k^2}.$$

So, f is Gateaux differentiable at $(0,0)$

with Gateaux derivative as

$$A \cdot h' = \frac{h^3}{h^2+k^2} \quad (\text{where } h' = (h,k))$$

but f is not Frechet differentiable.
as we know if f is Frechet differentiable then $\exists$ A bdd linear map such that

$$\lim_{h\to 0} \frac{|f(0+h)-f(0)-A\cdot h|_y}{\|h\|_x}=0$$

Take sequence $h_n' = \frac{1}{n}(1,0)$

$h_n' \to (0,0)$ as $n\to\infty$

$$\lim_{n\to\infty} \frac{|f(0+h_n')-f(0)-A\cdot h_n'|}{\|h_n'\|_{\mathbb{R}^2}}$$

$$= \lim_{n\to\infty} \frac{|f(\frac{1}{n}(1,0))-0-A\cdot h_n'|}{1/n}$$

$$= \lim_{n\to\infty} \frac{|f(1/n, 0)-A\cdot h_n'|}{1/n}$$

$$= \lim_{n\to\infty} \frac{\left|\frac{1/n^3}{1/n^2}-A\cdot h_n'\right|}{1/n}$$

$$= \lim_{n\to\infty} \left|1-\frac{A\cdot h_n'}{1/n}\right| = \lim_{n\to\infty}\left|1-A\left(\frac{h_n'}{1/n}\right)\right|$$

$$= \lim_{n\to\infty}\left|1-A\left(\frac{1/n\,(1,0)}{1/n}\right)\right|$$

$$= |1-A(1,0)|$$

$$\Rightarrow A(1,0)=1$$

Now take $h_n^2 = \frac{1}{n}(0,1)$

$h_n^2 \to 0$ as $n \to \infty$.

$$\lim_{n\to\infty} \frac{|f(0+h_n^2) - f(0) - A \cdot h_n^2|}{1/n}$$

$$= \lim_{n\to\infty} \frac{|f(0, 1/n) - 0 - A \cdot h_n^2|}{1/n}$$

$$= \lim_{n\to\infty} \left|0 - A \cdot \frac{h_n^2}{1/n}\right|$$

$$= \lim_{n\to\infty} \left|0 - A\left(\frac{\frac{1}{n}(0,1)}{1/n}\right)\right|$$

$$= |0 - A(0,1)|$$

$$\Rightarrow A(0,1) = 0$$

Now, take $h_n^3 = \frac{1}{n}(1,1)$

$$\lim_{n\to\infty} \frac{|f(0+h_n^3) - f(0) - A \cdot h_n^3|}{1/n}$$

$$= \lim_{n\to\infty} \frac{|f(1/n, 1/n) - 0 - A\left(\frac{1}{n}(1,1)\right)|}{1/n}$$

$$= \lim_{n\to\infty} \frac{\left|\frac{1/n^3}{1/n^2 + 1/n^2} - A\left(\frac{1}{n}(1,1)\right)\right|}{1/n}$$

$$= \lim_{n\to\infty} \left| \frac{1/n^3}{2/n^3} - \frac{A\cdot(1/n\,(1,1))}{1/n} \right|$$

$$= \left| \frac{1}{2} - A.(1,1) \right|$$

$\Rightarrow \quad A(1,1) = 1/2.$

A is not linear
as

$$A(1,1) = 1/2 \neq A(1,0) + A(0,1).$$

So, $\nexists$ any bdd linear map A from $\mathbb{R}^2$ to $\mathbb{R}$
Hence, f is not Frechet differentiable
at $(0,0)$.

Question-24 Prove that in an inner-product space the function $f(x) = \|x\|^2$ and $g(x) = \langle a, x \rangle$ are differentiable. Give formulas for the derivatives.

Proof :- Define, $f : X \to \mathbb{R}$

by $f(x) = \|x\|^2$

where X is an inner product space.

fix $x_0 \in X$.

Consider,

$$f(x_0+h) - f(x_0) = \|x_0+h\|^2 - \|x_0\|^2$$

$$= \langle x_0+h, x_0+h \rangle - \langle x_0, x_0 \rangle$$

$$= \langle x_0, h \rangle + \langle x_0, x_0 \rangle + \langle h, x_0 \rangle + \langle h, h \rangle - \langle x_0, x_0 \rangle$$

$$= \langle x_0, h \rangle + \langle h, x_0 \rangle + \langle h, h \rangle$$

$$= 2\langle h, x_0 \rangle + \langle h, h \rangle$$

Set $A.h = 2\langle h, x_0 \rangle$.

$$\frac{|f(x_0+h) - f(x_0) - A.h|}{\|h\|_X} = \frac{|\langle h, h \rangle|}{\|h\|_X}$$

$$= \frac{\|h\|_X^2}{\|h\|_X} = \|h\|$$

$$\lim_{h \to 0} \frac{|f(x_0+h) - f(x_0) - Ah|}{\|h\|_X} = 0.$$

Now, to check A is bdd linear map from X to $\mathbb{R}$.

A is linear since, inner product in the first argument is linear.

A is bdd.

$|Ah| = |2\langle h, x_0\rangle| = 2|\langle h, x_0\rangle|$
$\leq 2\|h\|\|x_0\|$
(by Cauchy-Schwarz inequality)

$= 2\|x_0\|\|h\|.$

So, $|Ah| \leq \lambda\|h\|$ (where $\lambda = 2\|x_0\|$ as x_0 is fixed)

Hence, A is bdd.
So, f is Frechet differentiable at x_0 with Frechet derivative A defined as
$A: X \to \mathbb{R}$
$A.h = 2\langle h, x_0\rangle.$

ii) ~~Let~~ Define $g: X \to \mathbb{R}$
as $g(x) = \langle a, x\rangle.$

fix $x_0 \in X$.
Consider,

$g(x_0+h) - g(x_0) = \langle a, x_0+h\rangle - \langle a, x_0\rangle$
$= \cancel{\langle a, x_0\rangle} + \langle a, h\rangle - \cancel{\langle a, x_0\rangle}$
$= \langle a, h\rangle.$

Set $A.h = \langle a, h\rangle$

$$\lim_{h\to 0} \frac{|g(x_0+h) - g(x_0) - A.h|}{\|h\|_X}.$$

$$= \lim_{h\to 0} \frac{0}{\|h\|_X} = 0.$$

Now, to check A is bdd linear map from X to $\mathbb{R}$.

A is linear

Since, X is real inner product space
So, it is linear in both argument.

A is bdd

$|Ah| = |\langle a, h\rangle| \leq \|a\| \|h\|$.
(by Cauchy-Schwarz inequality)

Hence, A is bdd.
So, f is Frechet differentiable at x_0 with Frechet derivative A defined as

$$A : X \longrightarrow \mathbb{R}$$

$$A.h = \langle a, h\rangle.$$