R04:

Department of Mathematics
MPhil/PhD Coursework Examination
Topics in Analysis

(July 2022)
Time: 3 hr
Max. Marks : 70

Attempt SEVEN questions in all. All symbols carry their usual meaning.

1. Let $K : [a, b] \times [a, b] \to \mathbb{R}$ be continuous and let $h \in C[a, b]$ be fixed. Find conditions on $\lambda \in \mathbb{R}$ such that the integral equation $x(t) = \lambda \int_a^t K(t, \tau)x(\tau)\, d\tau + h(t)$ has a unique solution in $x \in C[a, b]$. (10)

2. State Banach's Contraction Principle and Browder's Fixed point Theorem. Let $X = c_0$, the Banach Space of all real sequences that tend to zero, equipped with norm $\|x\| := \max_i |x_i|$ and suppose that the mapping f is given by $f(x) = (0, 1, x_1, x_2, x_3, \dots)$ where $x = (x_1, x_2, \dots) \in X$. Check if f is a (i) contraction or (ii) just a non-expansive map or neither. Further, verify if the conclusions of the above two theorems hold for f and if not, give reasons, justifying your claims in detail. (2+2+6)

3. (a) Let X, Y be a Banach spaces and $f : [a, b] \to X$ be a Riemann integrable function. If $A : X \to Y$ is a bounded linear operator then show that Af is also Riemann integrable and $A \int_a^b f(t)\, dt = \int_a^b Af(t)\, dt$. (4)

   (b) Let (M, Σ, μ) be measure space and X a Banach space. When is a function $f : M \to X$ said to be strongly measurable ? When is a strongly measurable function $f : M \to X$ Bochner measurable and how is its Bochner integral defined? Show that the definition of Bochner integral of f is independent of the choice of sequence of step functions. (6)

4. State Stone-Weierstrass's Theorem for an algebra of complex continuous functions defined on a compact set K. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{A}$ be the set of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ where $N \in \mathbb{N}, c_n \in \mathbb{C}, \theta \in \mathbb{R}$. Determine which of the conditions of the above mentioned result are satisfied by $\mathcal{A}$. Further check whether or not the conclusion of the theorem holds? Give a reason in either case. (2+8)

Similar Ex-17

5. (a) Let $F : C[0, 1] \to C[0, 1]$ given by $(F(\phi))(t) = \left[\int_0^t \phi(s)\, ds\right]^2$. Find the Frechet derivative of F at any $\phi_0 \in C[0, 1]$. (6)

Ex-6

   (b) Let A be a bounded linear operator on a real Hilbert space H and define $F : H \to \mathbb{R}$ by $F(x) = \langle x, Ax \rangle$. Find the Frechet derivative of F at $x_0 \in H$. (4)

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6. (a) Let f be defined on an open set Ω in the direct sum space $X = X_1 \oplus X_2 \oplus X_3$, (where each X_j is a Banach space), and take values in a normed space Y, such that all partial derivatives $D_j f$ exist in Ω and are continuous at every point x_0 in Ω. Show that f is Frechet differentiable at x_0 and find an expression for the Frechet derivative of f at x_0. (5)

$\|f(b)-f(a)\| \leq \|b-a\| \sup f'(\cdot)$

Ex-3-7
Q-2

MVT-III

$\|a+b\| \leq \|a\|+\|b\|$

$|\,|a|-|b|\,| \leq \|a-b\|$

(b) Let $f : \Omega \to Y$ be Frechet differentiable, where $\Omega \subset X$, is open, X, Y are open and suppose $a, b, x_0 \in \Omega$ with the line segment S joining a to b contained in Ω. Show that $\|f(b) - f(a) - f'(x_0)(b - a)\| \leq \|b - a\| \sup_{x \in S} \|f'(x) - f'(x_0)\|$. (5)

7. State and prove a Surjective Mapping Theorem for a function $f : \Omega \to Y$ where $\Omega \subset X$ is an open set and X, Y are Banach spaces. You may clearly state and use an appropriate implicit function theorem if required. (10)

8. (a) Show that for $f \in L^p(\mathbb{R}^N), g \in L^1(\mathbb{R}^N), 1 \leq p \leq \infty$ the function $y \mapsto f(x - y)g(y)$ is integrable for almost every $x \in \mathbb{R}^N$. (4)

   (b) Define convolution $f * g$ for $f \in L^p(\mathbb{R}^N), g \in L^1(\mathbb{R}^N), 1 \leq p < \infty$. Show further that $f * g \in L^p(\mathbb{R}^N)$ and $\|f * g\|_p \leq \|f\|_p \|g\|_1$. (6)

9. (a) Let $\Omega \subset \mathbb{R}^N$ be open. When is a function $f \in L^1_{loc}(\Omega)$ said to be weakly differentiable? Show that for $\Omega = (-2, 2)$ the function $f(t) = |t|$ is weakly differentiable and find its weak derivative. (4)

   (b) Show that if $f \in L^1_{loc}(\Omega)$ is weakly differentiable with weak derivative of f zero, then $f =$ constant. (6)

10. Let $\Omega = (a, b) \subset \mathbb{R}$. Define the spaces $W^{1,p}(a, b), 1 \leq < \infty$. Show that these spaces are Banach spaces with respect to appropriate norms. Further, show that $W^{1,p}(a, b) \subset C[a, b]$ if $a, b \in \mathbb{R}$. (10)