Department of Mathematics
MPhil/PhD Coursework Examination
Topics in Analysis

(March 2020)

Time: 3 hrs. Max. Marks : 70

Attempt FIVE questions in all. All symbols carry their usual meaning.

1. (a) Let F be a mapping from a Hilbert space H into itself such that $\langle Fx - Fy, x - y\rangle \geq \alpha\|x-y\|^2$ $(\alpha > 0)$ and $\|Fx - Fy\| \leq \beta\|x-y\|$. Show that for an appropriate $\lambda > 0$, the map $Gx = x - \lambda(Fx - w)$ has a fixed point. Is it unique? Hence deduce that F is bijective. (6)

   (b) Let $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous mapping satisfying $|f(s, t_1) - f(s, t_2)| \leq \lambda|t_1 - t_2|$ where λ is a constant depending only on f. Show that the initial-value problem $x'(s) = f(s, x(s)), x(0) = \beta$ has a unique solution in $C[0, b]$. Can the above result be used to show that the equation $x'(s) = sin(x(s)e^s), x(0) = \beta$ has a unique solution in $C[0, 4]$? Justify (6+2)

2. (a) State Brouwer's Fixed Point Theorem in $\mathbb{R}^n$. Give an example to show that this result does not hold in infinite dimensions. (4)

   (b) State Browder's fixed point Theorem. Let $X = l^2$ with its usual norm and the mapping f be given by $f(x) = (1, x_1, x_2, x_3, \dots)$ where $x = (x_1, x_2, \dots) \in l^2$. Show that F has no fixed points and state, with justification, why Browder's Theorem does not yield a fixed point in this case? (1+4)

   (c) Is the function $f : (0, 1) \to L^\infty(0, 1)$ Bochner integrable when (a) $f(t) := \chi_{(0,t)}$ (b) $f(t) := tx_0$ where x_0 is a fixed element of $L^\infty(0, 1)$. Justify. (3+2)

3. (a) Let X be a topological space and $C(X)$ the vector space of all bounded continuous complex-valued functions on X. When is a set $S \subset C(X)$ said to separate points? Illustrate with a non-trivial example. (2)

   (b) State the real and complex Stone-Weierstrass Theorems. Prove the Complex Stone-Weierstrass Theorem, assuming the real one to hold. (2 + 4)

   (c) For a mapping $f : D \to X$ where X is a normed space and D an open subset of X, explain the difference between the statements (a) f' is continuous at x (b) $f'(x)$ is continuous. Prove that if $f'(x)$ exists, then it is continuous and differentiable. (6).

4. (a) Let $g \in C[0, 1], \alpha \in \mathbb{C}$ be fixed. Define $F : C[0, 1] \to C[0, 1]$ by $F(x) = \alpha g \cdot x, x \in C[0, 1]$ where $\cdot$ represents pointwise multiplication. Find the derivative of F at any $x_0 \in C[0, 1]$. (4)

   (b) Let $f : X \to X$ be differentiable, X a real Hilbert space, and $v \in X$. Define $g : X \to \mathbb{R}$ by $g(x) = \langle f(x), v\rangle$. Prove that g is differentiable and find its derivative. (5)

   (c) Let $f : H \to \mathbb{R}$ where H is a Hilbert space be given by $f(x) := \langle x, x\rangle, x \in H$. Show that f is differentiable and find its derivative. (5)

5. (a) What is meant by spherical distance? Compute the expression for the spherical distance $|z_1, z_2|$ between any two points z_1, z_2 of the extended complex plane? (1 + 4)

   (b) Show that if f is a meromorphic function in a domain D then it is continuous in D with respect to the spherical distance. (4)

   (c) What is meant by a C_0 sequence in a domain D? Illustrate with a non-trivial example. Further, show that a C_0 sequence of meromorphic functions has a limit function F. What can be said about the function F? Give statements only. (5)

6. (a) Show that a family of normal holomorphic functions which is locally uniformly bounded in a domain D is normal in D. Check whether or not the family $\{f_\epsilon : 0 < \epsilon \leq 1/2\}$ where $f_\epsilon(z) = \frac{z}{(2z+\epsilon)}$ is normal on the disc of radius 1 centered at 2? (3+3)

   (b) Define spherical derivative at a point $z_0 \in D$ of a meromorphic function defined on a domain D. Illustrate this definition with a non-trivial example. State and prove a characterisation for normality of a family of meromorphic maps in terms of the spherical derivatives of its members. (1+1+6)

7. (a) Let $\mathcal{F}$ be a family of meromorphic functions defined on a domain D and a, b, c be three distinct values in $\hat{\mathbb{C}}$ such that each of the functions $f(z) = a, f(z) = b, f(z) = c$ has no root in D. Prove that the family $\mathcal{F}$ is normal.
   (You may assume (and state) a suitable result about families of holomorphic functions.) (5)

   (b) Show that the family $\{f_n\}$ given by $f_n(z) = f(2^n z), n = 1, 2, \ldots$ is not normal in the disk $\{\lambda \in \mathbb{C} : |\lambda| < 2\}$ whenever f is a non constant meromorphic function. (5)

   (c) Let $f_n(z) := 3^n z + 3^{2n} z^2$ and $g_n(z) = \frac{z}{(z - \frac{1}{3^n})}$. What can you deduce about the normality of the families $\{f_n\}$ and $\{g_n\}$ in the domain $D = \{z : |z| < 3\}$. Justify. (2+2)