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$\boxed{3.1}$

Que. 18 Prove that if f is differentiable at x, then f is Lipschitz continuous at x. This means that $\| f(y) - f(x) \| \le \lambda \| y - x \|$ for some λ and all y in a neighborhood of x.

$f : X \to Y$ (X and Y are normed spaces).

f is differentiable at x, i.e.

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - A(h) \|_Y}{\| h \|_X} = 0$$

where $A : X \to Y$ is linear, bounded f^n.

Let $\epsilon > 0$, then $\exists$ $\delta > 0$ such that

$$\frac{\| f(x+h) - f(x) - A(h) \|}{\| h \|} \le \epsilon \quad \text{whenever } \|h\| < \delta \qquad - (1)$$

Let $x \in X$

Any nbd. of x is a ball in X.

let $B(x;\delta)$ be a nbd. of x.

$$B(x;\delta) = \{ z \mid \|x - z\| < \delta \}$$

let $y = x + h$ then $\because$ $\| x - y \| = \| h \| < \delta$

$\Rightarrow y \in B(x;\delta)$.

Consider $\|f(y) - f(x)\| = \|f(x+h) - f(x) + A(h) - A(h)\|$

$\leq \|f(x+h) - f(x) - A(h)\| + \|A(h)\|$

(By Triangle's Ineq.)

$\leq \epsilon \|h\| + \|A\| \cdot \|h\|$

(By ① and $\because$ A is bdd. $\Rightarrow \|A(h)\| \leq \|A\| \|h\|$)

$= (\epsilon + \|A\|) \|h\|$

$\because$ A is bdd. $\Rightarrow$ $\|A\|$ is some finite no.

Putting $\epsilon + \|A\| = d$, we get

$\|f(y) - f(x)\| \leq d \|y - x\|$

$\because$ y was arbitrary $\therefore$ true for any y in a nbd. of x.

Que 19. let a_n $(n = 0, 1, 2, \ldots)$ be real nos. such that $\sum_{n=0}^{\infty} a_n z^n$ converges $\forall z \in \mathbb{C}$. let X be a Banach space. Define $f : \mathcal{L}(X, X) \to \mathcal{L}(X, X)$ by eq^n. $f(A) = \sum_{n=0}^{\infty} a_n A^n$. What is frechet derivative of f ?

let's first check if $f(A) \in \mathcal{L}(X, X)$.

i.e. $\sum_{n=0}^{\infty} a_n A^n$ is a bounded and linear operator

Linear

let $s, t \in X$, $\alpha \in \mathbb{C}$.

$$\sum_{n=0}^{\infty} a_n A^n (t + \alpha s) = \lim_{k \to \infty} \sum_{n=0}^{k} a_n A^n (t + \alpha s)$$

$$= \lim_{k\to\infty} \left(a_0 (t+\alpha s) + a_1 A(t+\alpha s) + \cdots + a_k A^k (t+\alpha s) \right)$$

$$= \lim_{k\to\infty} \left(a_0 t + a_0 \alpha s + a_1 A(t) + a_1 \alpha A(s) + \cdots + a_k A^k(t) + a_k \alpha A^k(s) \right)$$

$$(\because A \text{ is linear})$$

$$= \lim_{k\to\infty} \left(\sum_{n=0}^{k} a_n A^n(t) + \alpha \sum_{n=0}^{k} a_n A^n(s) \right)$$

$$= \sum_{n=0}^{\infty} a_n A^n(t) + \alpha \sum_{n=0}^{\infty} a_n A^n(s)$$

$$\therefore \sum_{n=0}^{\infty} a_n A^n (t+\alpha s) = \sum_{n=0}^{\infty} a_n A^n(t) + \alpha \sum_{n=0}^{\infty} a_n A^n(s)$$

$$\therefore \sum_{n=0}^{\infty} a_n A^n \text{ is linear}$$

Bounded Let $t \in X$.

$$\left\| \sum_{n=0}^{\infty} a_n A^n \right\| = \sup_{\|t\|\neq 0} \frac{\left\| \sum_{n=0}^{\infty} a_n A^n(t) \right\|}{\|t\|}$$

$$= \sup_{\|t\|\neq 0} \frac{\left\| \lim_{k\to\infty} \sum_{n=0}^{k} a_n A^n(t) \right\|}{\|t\|}$$

$$= \sup_{\|t\|\neq 0} \lim_{k\to\infty} \frac{\left\| \sum_{n=0}^{k} a_n A^n(t) \right\|}{\|t\|}$$

We can interchange limit and norm as, $\left\{ \sum_{n=0}^{k} a_n A^n(t) \right\}_k$ is convergent, since —

we have $\sum_{n=0}^{\infty} |a_n| \, \|A\|^n$ converges $\left(\text{Given } \sum_{n=0}^{\infty} a_n z^n \text{ converges} \Rightarrow \sum_{n=0}^{\infty} |a_n| z^n \text{ converges} \right.$

$\because$ Series cvgs $\Leftrightarrow$ seq^n. of partial sums is a Cauchy sequence.

let $\epsilon' > 0$ then $\exists\, n_0 \in \mathbb{N} \ni \forall\, M \geq N \geq n_0$

we have $\sum_{n=N+1}^{M} |a_n| \, \|A\|^n < \epsilon'$ — (1)

(Series is in $\mathbb{R}$ and $\because$ every term is positive so we can remove modulus.)

Now, to prove $\left\{ \sum_{n=0}^{k} a_n A^n(t) \right\}_k$ cvgs.

$\because$ X is Banach, being Cauchy suffice.

let $\epsilon > 0$ and $\epsilon = \epsilon' . \|t\|$

for above M, N consider,

$$\left\| \sum_{n=0}^{M} a_n A^n(t) - \sum_{n=0}^{N} a_n A^n(t) \right\| = \left\| \sum_{n=N+1}^{M} a_n A^n(t) \right\|$$

$$\leq \sum_{n=N+1}^{M} |a_n| \|A^n(t)\| \quad \text{(by Triangle's Ineq.)}$$

$$\leq \sum_{n=N+1}^{M} |a_n| \, \|A\|^n \, \|t\| \quad (\because A \text{ is bdd.})$$

$$= \|t\| \sum_{n=N+1}^{M} |a_n| \, \|A\|^n$$

$$\leq \underbrace{\|t\| \, \epsilon'}_{=\epsilon} \quad \text{by (1)}$$

$$\Rightarrow \left\| \sum_{n=0}^{M} a_n A^n(t) - \sum_{n=0}^{N} a_n A^n(t) \right\| \leq \epsilon$$

$\Rightarrow$ Cauchy. Hence, $\left\{ \sum_{n=0}^{k} a_n A^n(t) \right\}_k$ cvgs.

So, $$\left\| \sum_{n=0}^{\infty} a_n A^n \right\| = \sup_{\|t\| \neq 0} \lim_{k \to \infty} \frac{\left\| \sum_{n=0}^{k} a_n A^n(t) \right\|}{\|t\|}$$

Also, $\left\| \sum_{n=0}^{k} a_n A^n(t) \right\| \leq \sum_{n=0}^{k} |a_n| \, \|A\|^n \, \|t\|$ as done above

thus, we get $$\left\| \sum_{n=0}^{\infty} a_n A^n \right\| \leq \sup_{\|t\| \neq 0} \lim_{k \to \infty} \frac{\sum_{n=0}^{k} |a_n| \, \|A\|^n \|t\|}{\|t\|}$$

$$\leq \sup_{\|t\| \neq 0} \lim_{k \to \infty} \sum_{n=0}^{k} |a_n| \, \|A\|^n$$

$$= \sup_{\|t\| \neq 0} \sum_{n=0}^{\infty} |a_n| \, \|A\|^n$$

$$= \sum_{n=0}^{\infty} |a_n| \, \|A\|^n$$

$\because$ this series is cgt. $\therefore \sum_{n=0}^{\infty} |a_n| \, \|A\|^n = l$ (say)

where l is its limit & is a finite no.

$$\therefore \quad \left\| \sum_{n=0}^{\infty} a_n A^n \right\| \leq l$$

$\Rightarrow \sum_{n=0}^{\infty} a_n A^n$ is bounded.

<u>Claim</u> :- Frechet der. of f, say, $B : \mathcal{L}(X, X) \to \mathcal{L}(X, X)$

is defined as :-

$$B(H) = \sum_{n=1}^{\infty} a_n \left(A^{n-1} H + A^{n-2} H A + A^{n-3} H A^2 + \cdots + A H A^{n-2} + H A^{n-1} \right)$$

So, we need to show :-

(i) $\dfrac{\| f(A+H) - f(A) - B(H) \|}{\|H\|} \to 0$ as $\|H\| \to 0$

(ii) B is a linear, bounded function from $\mathcal{L}(X, X) \to \mathcal{L}(X, X)$.

(i) Firstly, consider $\| f(A+H) - f(A) - B(H) \|$

$$= \left\| \sum_{n=0}^{\infty} a_n (A+H)^n - \sum_{n=0}^{\infty} a_n A^n - \sum_{n=1}^{\infty} a_n \left(A^{n-1} H + A^{n-2} H A + A^{n-3} H A^2 + \cdots + A H A^{n-2} + H A^{n-1} \right) \right\|$$

$$\left(\begin{array}{ll} \text{for } n=0. \text{ we have}, & a_0 - a_0 = 0 \\ \text{for } n=1 , & a_1(A+H) - a_1 A - a_1 H = 0. \end{array} \right)$$

$$= \left\| \sum_{n=2}^{\infty} a_n \left[(A+H)^n - A^n - (A^{n-1}H + A^{n-2}HA + \cdots + HA^{n-1}) \right] \right\|$$

$$\le \sum_{n=2}^{\infty} \left\| a_n \left[(A+H)^n - A^n - (A^{n-1}H + \cdots + HA^{n-1}) \right] \right\|$$

$$\le \sum_{n=2}^{\infty} |a_n| \left\| (A+H)^n - A^n - (A^{n-1}H + \cdots + HA^{n-1}) \right\|$$

using
$$(A+H)^n = A^n + A^{n-1}H + A^{n-2}HA + \cdots + HA^{n-1} + A^{n-2}H^2 + A^{n-3}H^2A + \cdots + H^2A^{n-2} + A^{n-3}HAH + A^{n-4}HAHA + \cdots + HAHA^{n-3} + A^{n-4}HA^2H + \cdots + HA^2HA^{n-4} + \cdots + A^{n-3}H^3 + A^{n-4}H^3A + \cdots H^3A^{n-3} + \cdots\cdots + AH^{n-1} + HAH^{n-2} + \cdots + H^{n-1}A + H^n.$$

we get,
$$\le \sum_{n=2}^{\infty} |a_n| \left\| A^{n-2}H^2 + A^{n-3}H^2A + \cdots + H^2A^{n-2} + A^{n-3}HAH + \cdots + HAHA^{n-3} + \cdots + AH^{n-1} + \cdots + H^{n-1}A + H^n \right\|$$

$$\le \sum_{n=2}^{\infty} |a_n| \left(\|A^{n-2}H^2\| + \|A^{n-3}H^2A\| + \cdots + \|H^{n-1}A\| + \|H^n\| \right)$$

$\because$ A and H are bdd. operators, $\therefore$, 4.

$$\leq \sum_{n=2}^{\infty} |a_n| \left(\|A\|^{n-2} \|H\|^2 + \dots + \|H\|^{n-1} \|A\| + \|H\|^n \right)$$

$$\leq \sum_{n=2}^{\infty} |a_n| \, \|H\|^2 \left(\|A\|^{n-2} + \|A\|^{n-3} \|A\| + \dots + \|H\|^{n-3} \|A\| + \|H\|^{n-2} \right)$$

$$\Rightarrow \frac{\| f(A+H) - f(A) - B(H) \|}{\|H\|} \leq \|H\| \sum_{n=2}^{\infty} |a_n| \left(\|A\|^{n-2} + \dots + \|H\|^{n-2} \right)$$

$$\longrightarrow 0 \quad \text{as } \|H\| \longrightarrow 0.$$

($\because$ we are given $\sum_{n=0}^{\infty} a_n z^n$ crgs. for every $z \in \mathbb{C}$. $\therefore \Rightarrow \sum_{n=0}^{\infty} |a_n| z^n$ crgs. $(\|A\|^{n-2} + \dots + \|H\|^{n-2})$ are real nos. $\therefore$ this series crgs. & hence bounded)

(ii.) B is linear :—

let $H, G \in \mathcal{L}(X, X)$.

$$B(G + \alpha H) = \sum_{n=1}^{\infty} a_n \left(A^{n-1} (G + \alpha H) + A^{n-2} (G + \alpha H) A + \dots + (G + \alpha H) A^{n-1} \right)$$

$$= \lim_{k \to \infty} \sum_{n=1}^{k} a_n \left[A^{n-1} G + \alpha A^{n-1} H + A^{n-2} G A + \alpha A^{n-2} H A + \dots + G A^{n-1} + \alpha H A^{n-1} \right]$$

$$= \lim_{k \to \infty} \left(\sum_{n=1}^{k} a_n \left[A^{n-1} G + A^{n-2} G A + \dots + G A^{n-1} \right] + \sum_{n=1}^{k} \alpha a_n \left[A^{n-1} H + A^{n-2} H A + \dots + H A^{n-1} \right] \right)$$

$$= \sum_{n=1}^{\infty} a_n \left[A^{n-1} G + \dots + G A^{n-1} \right] + \alpha \sum_{n=1}^{\infty} a_n \left[A^{n-1} H + \dots + H A^{n-1} \right]$$

$$= B(G) + \alpha B(H)$$

$\therefore$ linear.

B is bounded :—

$$\|B\| = \sup_{\|H\| \neq 0} \frac{\left\| \sum_{n=1}^{\infty} a_n \left(A^{n-1}H + A^{n-2}HA + \dots + HA^{n-1} \right) \right\|}{\|H\|}$$

$$= \sup_{\|H\| \neq 0} \frac{\left\| \lim_{k \to \infty} \sum_{n=1}^{k} a_n \left(A^{n-1}H + A^{n-2}HA + \dots + HA^{n-1} \right) \right\|}{\|H\|}$$

$$= \sup_{\|H\| \neq 0} \lim_{k \to \infty} \frac{\left\| \sum_{n=1}^{k} a_n \left(A^{n-1}H + A^{n-2}HA + \dots + HA^{n-1} \right) \right\|}{\|H\|}$$

$\because$ this seq^n. will cvg. (we can prove by same steps as we did before)
$\therefore$ we can interchange norm & lim.

Now, $\left\| \sum_{n=1}^{k} a_n \left(A^{n-1}H + A^{n-2}HA + \dots + HA^{n-1} \right) \right\|$

$$\leq \sum_{n=1}^{k} |a_n| \left(\|A\|^{n-1} \|H\| + \|A\|^{n-2} \|H\| \|A\| + \dots + \|H\| \|A\|^{n-1} \right)$$

(By triangle's ineq. & $\because$ A & H are bdd. op.)

$$= \|H\| \sum_{n=1}^{k} |a_n| \left(\|A\|^{n-1} + \dots + \|A\|^{n-1} \right)$$

$$\Rightarrow \|B\| \leq \sup_{\|H\| \neq 0} \lim_{k \to \infty} \frac{\|H\| \sum_{n=1}^{k} |a_n| \left(\|A\|^{n-1} + \dots + \|A\|^{n-1} \right)}{\|H\|}$$

$$= \sup_{\|H\| \neq 0} \sum_{n=1}^{\infty} |a_n| \, n \, \|A\|^{n-1}.$$

Again, $\sum_{n=1}^{\infty} |a_n| (n \|A\|^{n-1})$ will converge

$\Rightarrow \sum_{n=1}^{\infty} |a_n| (n \|A\|^{n-1}) = M$ for some M

$\Rightarrow \|B\| \leq M$

$\Rightarrow B$ is bdd.

Also, $B : \mathcal{L}(X, X) \longrightarrow \mathcal{L}(X, X)$

We can show $\sum_{n=1}^{\infty} a_n (A^{n-1} H + A^{n-2} HA + \cdots + HA^{n-1})$ is linear and bdd. as we have shown $\sum_{n=0}^{\infty} a_n A^n$.

Hence $B(H)$ is frechet derivative of $f(A) = \sum_{n=0}^{\infty} a_n A^n$ given by

$$B(H) = \sum_{n=1}^{\infty} a_n (A^{n-1} H + A^{n-2} HA + \cdots + HA^{n-1}).$$