To show that $\int_M f(t)\,dt$ is independent of the choice of step function.

<u>Proof</u> Let $\{s_m\}_{m=1}^{\infty}$ and $\{\alpha_m\}_{m=1}^{\infty}$ be two sequence of step functions st

$$\lim_{m\to\infty} s_m(t) = f(t) \quad a.e$$

$$\lim_{m\to\infty} \alpha_m(t) = f(t) \quad a.e.$$

and

$$\lim_{m\to\infty} \int \| s_m(t) - f(t) \|_X \, d\mu = 0 \quad \text{--- (1)}$$

$$\lim_{m\to\infty} \int \| \alpha_m(t) - f(t) \|_X \, d\mu = 0 \quad \text{--- (2)}$$

To Show :-

$$\lim_{m\to\infty} \int_M s_m(t)\,d\mu = \lim_{m\to\infty} \int_M \alpha_m(t)\,d\mu$$

Consider

$$\left\| \int_M s_m(t)\,d\mu - \int_M \alpha_m(t)\,d\mu \right\|_X$$

$$= \left\| \int_M (s_m(t) - \alpha_m(t))\,d\mu \right\|_X$$

Now as $s_m(t)$ and $\alpha_m(t)$ are step functions so $s_m(t) - \alpha_m(t)$ is also a step function.

therefore we have

$$0 \le \lim_{m\to\infty} \left\| \int_M (s_m(t) - \lambda_m(t))\, d\mu \right\|_X \le \lim_{m\to\infty} \int_M \| s_m(t) - \lambda_m(t) \|_X \, d\mu$$

$$\le \lim_{m\to\infty} \int_M \| s_m(t) - f(t) + f(t) - \lambda_m(t) \|_X \, d\mu$$

$$\le \lim_{m\to\infty} \int_M \left(\| s_m(t) - f(t) \|_X + \| f(t) - \lambda_m(t) \|_X \right) d\mu$$

$$\le \lim_{m\to\infty} \int_M \| s_m(t) - f(t) \|_X \, d\mu + \lim_{m\to\infty} \int_M \| f(t) - \lambda_m(t) \|_X \, d\mu$$

from ① and ② we have

$$\lim_{m\to\infty} \left\| \int_M (s_m(t) - \lambda_m(t))\, d\mu \right\|_X = 0$$

$$\Rightarrow \left\| \lim_{m\to\infty} \int_M (s_m(t) - \lambda_m(t))\, d\mu \right\|_X = 0$$

$$\Rightarrow \lim_{m\to\infty} \int_M (s_m(t) - \lambda_m(t))\, d\mu = 0$$

$$\Rightarrow \lim_{m\to\infty} \int_M s_m(t)\, d\mu - \lim_{m\to\infty} \int_M \lambda_m(t)\, d\mu = 0$$

$$\Rightarrow \lim_{m\to\infty} \int_M s_m(t)\, d\mu = \lim_{m\to\infty} \int_M \lambda_m(t)\, d\mu$$

let s be a step function from $S: M \to X$ where M is a measure space and X is a Banach space then we have

$$\left\| \int_M s(t)\, d\mu \right\|_X \leq \int_M \| s(t) \|_X \, d\mu$$

Proof

Consider

$$\int_M s(t)\, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i)$$

$$\left\| \int_M s(t)\, d\mu \right\|_X = \left\| \sum_{i=1}^{n} a_i \, \mu(A_i) \right\|$$

$$\left\| \int_M s(t)\, d\mu \right\|_X \leq \sum_{i=1}^{n} |\mu(A_i)| \, \| a_i \| \quad \text{—} (1)$$

Now consider

$$\phi(t) = \| s(t) \|_X$$

$$\phi(t): M \to \mathbb{R}$$

now we will prove that $\phi(t)$ is also a step function.

As $s(t)$ is a step function so $\exists$ $A_1, A_2, \ldots A_n$, pair wise disjoint, $A_i \in \Sigma$ with $\mu(A_i) < \infty$

$$s(t) = \sum_{i=1}^{n} a_i \, \chi_{A_i}(t)$$

when $t \in A_1$

$$s(t) = a_1$$

$$\phi(t) = \|a_1\|$$

$$\phi(t) = \|a_1\| \chi_{A_1} \qquad \forall\, t \in A_1$$

$\forall\, t \in A_2$

$$s(t) = a_2$$

$$\phi(t) = \|a_2\|$$

$$\phi(t) = \|a_2\| \chi_{A_2}$$

$$\phi(t) = \|a_i\| \chi_{A_i}$$

when $t \in M - \cup A_i$

$$s(t) = 0$$

$$\phi(t) = 0$$

$$\phi(t) = \sum_{i=1}^{n} \|a_i\| \chi_{A_i}(t)$$

$\Rightarrow$ $\phi(t)$ is a step function

$$\int_M \phi(t)\, d\mu = \int_M \|s(t)\|_X\, d\mu = \sum \|a_i\| \chi_{A_i}(t) \quad \text{—②}$$

from ① and ②

$$\left\| \int_M s(t)\, d\mu \right\|_X \leq \int_M \|s(t)\|_X\, d\mu$$

for all step functions $s : M \to X$

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