Department of Mathematics
MPhil/PhD Coursework Examination
Topics in Analysis

Time: 3 hrs. (Jan 2018) Max. Marks : 50

Attempt FIVE questions in all. All symbols carry their usual meaning.

1. State and prove Banach's Fixed Point Theorem (or the Contraction Principle). Pick two hypotheses from the statement and show, with the help of examples, that these hypotheses cannot be weakened. (1+4+5)

2. State Browder's Fixed Point Theorem.
Let $X = C[0,1]$ with its usual sup norm. Let $M = \{x \in X : 0 \le x(t) \le 1, x(0) = 0, x(1) = 1\}$ and the mapping F be given by $(Fx)(t) = tx(t), t \in [0,1]$. Show that (a) $F(M) \subset M$, (b) F is non-expansive and (c) F has no fixed point in M. Why does Browder's Theorem not yield a fixed point in this case? (2+7+1)

3. (a) State the Stone Weierstrass theorem. Also state the Weierstrass approximation theorem for continuous function defined on an interval $[a, b]$. Show that the latter may be considered as a special case of the former. (3)

(b) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{A}$ be the set of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ where $N \in \mathbb{N}, c_n \in \mathbb{C}, \theta \in \mathbb{R}$. Show that $\mathcal{A}$ is an algebra which separates points of $\mathbb{T}$ and vanishes at no point of $\mathbb{T}$. Using the fact that $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\, d\theta = 0$ for $f \in \mathcal{A}$ show that there are continuous functions on $\mathbb{T}$ which are not in the uniform closure of $\mathcal{A}$. Why is the Stone Weierstrass theorem not applicable here? (7)

4. Let $f : D \to Y$ be a mapping from an open subset D of a normed linear space X to a normed linear space Y and let $x \in D$. When is f said to be Frechet differentiable at x ?

(a) Show that the Frechet derivative, if it exists, is unique.

(b) If f, g are Frechet differentiable at x show that so is $f + g$ and $(f + g)'(x) = f'(x) + g'(x)$.

(c) Let $f : D \to Y$ be differentiable at $x \in D$ and Z be a normed linear space and $B : Y \to Z$ be a bounded linear operator. Prove that $B \circ f$ is Frechet differentiable at x and find $(B \circ f)'(x)$. (1+3+3+3)

5. (a) Let $y_0 \in Y$ where Y is a normed linear space. Define $f, g : \mathbb{R} \to Y$ by $f(t) = ty_0$ and $g(t) = (\sin t)y_0$. Compute the Frechet derivatives of f and g. (5)

(b) Let $X = C[0,1]$ with its usual sup norm and fix $n \in \mathbb{N}$. Select $t_i \in [0,1]$ and $v_i \in C[0,1]$ and set

$$F(x) = \sum_{i=1}^{n} |x(t_i)|^2 v_i.$$