

Lecture 4

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$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

and $x_0 \in \mathbb{R}^n$.

$$f = (f_1, f_2, \dots, f_m).$$

- $D_j f_i$ exists $\forall i$
at in a
nbd of x_0
 $i=1, 2, \dots, m$
 $j=1, \dots, n$

Then f is Fréchet Diff.
at x_0

02

$$f'(x_0) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

$$\cong M_{m \times n}(\mathbb{R})$$

$$f'(x_0) \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \begin{bmatrix} D_1 f_1 & \dots & D_n f_1 \\ \vdots & \ddots & \vdots \\ D_1 f_m & \dots & D_n f_m \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

$$= J h.$$

$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$

Jacobian $\left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Th. 1.10}$

Defn 1.11: Gâteaux diff. [03.]

Let $D \subseteq X$, be open, X, Y
normed spaces and
 $f: D \rightarrow Y$. We give
 f is stb Gâteaux diff
at $x_0 \in D$, if \exists a bdd
linear map A s.t

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \| f(x_0 + \lambda h) - f(x_0) - \lambda Ah \| = 0$$

for every $h \in X$.

• If f is Fréchet diff at x , then it is Gâteaux diff at x and the two derivatives are equal.

• Converse ???

give examples

Ex

Theorem 1.12 : Chain

Rule :

let $f : D \rightarrow Y, D \subseteq X$.

$g : E \rightarrow Z.$

E open; $f(D) \subseteq E$.

If f is differentiable at x_0 and if g is diff. at $f(x_0)$ then $g \circ f$ is diff. at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0)$$

Pf.: Let $F = g \circ f$.

$$A = f'(x_0) \in \mathcal{B}(X, Y)$$

$$B = g'(f(x_0)) \in \mathcal{B}(Y, Z)$$

We have to show that

$$F'(x_0) = BA$$

$$F(x_0+h) - F(x_0) - BA(h)$$

□

$$= g(f(x_0+h)) - g(f(x_0)) - BA(h)$$

$$= g(f(x_0+h) - f(x_0) - Ah + f(x_0) + Ah) - g(f(x_0)) - BA(h) \quad (1)$$

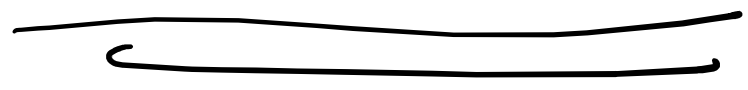
$$\text{def } o_1(h) = f(x_0+h) - f(x_0) - Ah.$$

$$\frac{o_1(h)}{\|h\|} \rightarrow 0 \text{ as } h \rightarrow 0 \quad (2)$$

$$\begin{aligned}
&= g\left(\overset{y}{f(x_0)} + Ah + o_1(h)\right) \\
&\quad - g(f(x_0)) \\
&\quad - BA h.
\end{aligned}$$

$$\begin{aligned}
&= g(y + \psi(h)) - g(y) \\
&\quad - BA h. \quad (3)
\end{aligned}$$

where $y = f(x_0)$
and $\psi(h) = Ah + o_1(h)$.



Log.

$$\underbrace{g(y + \Delta(h)) - g(y)} + B \Delta(h) - \underbrace{B \Delta(h)} - BA h.$$

$$= g(y) + B \Delta(h) + (g(y + \Delta(h)) - g(y) - B \Delta(h)) - g(y) - BA h.$$

$$= g(y) + B \Delta(h) + o_2(\Delta(h)) - g(y) - BA h.$$

where
 $O_2(\Delta(h))$

$$= g(y + \Delta(h)) - g(y) - B\Delta(h)$$

$$\frac{O_2(\Delta(h))}{\|\Delta(h)\|} \rightarrow 0 \quad \text{if} \quad \|\Delta(h)\| \rightarrow 0$$

Mean Value Theorem III.

Ex
1.13

$$f: \mathbb{R} \longrightarrow \mathbb{R}^2.$$

$$g: \mathbb{R} \longrightarrow \mathbb{R}^2.$$

$$f(t) = (t, \sin t)$$

$$g(t) = (\cos t, \sin t).$$

Consider $f, g: [0, 2\pi] \longrightarrow \mathbb{R}^2.$

at any t , $f'(t), g'(t)$

are 2×1 matrices. 112

$$f'(t) = \begin{bmatrix} 1 \\ \cos t \end{bmatrix}$$

$$g'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

Does $\exists t_0 \in (0, 2\pi)$ st

$$f(2\pi) - f(0) = 2\pi f'(t_0)$$

$$g(2\pi) - g(0) = 2\pi g'(t_0).$$

$$\begin{pmatrix} 2\pi, 0 \end{pmatrix}^T = 2\pi \begin{pmatrix} 1, \cos t_0 \end{pmatrix}^T.$$

$t_0 = \pi/2$ works?

$$g(2\pi) - g(0)$$

$$= (1, 0) - (1, 0)$$

$$= (0, 0)$$

$$RMS = \pi \begin{pmatrix} -\sin t_0 \\ \cos t_0 \end{pmatrix}$$

~~F~~ any t_0 at t_0
 $\sin t_0 = 0 = \cos t_0$

Th 1.14 .

14.

Mean Value Theorem I

Let $f: D \rightarrow \mathbb{R}$, where
 $D \subseteq X$ is open: let

$a, b \in D$. Assume

that the line segment

$$[a, b] = \left\{ a + t(b-a) : 0 \leq t \leq 1 \right\}$$

lies in D .

If f is continuous
on $[a, b]$ and diff

on the open line $\lfloor 15$
segment (a, b) , then
for some $\xi \in (a, b)$

$$f(b) - f(a)$$

$$= f'(\xi)(b-a) \dots$$

~~$$= (b-a)f'(\xi) \dots$$~~

Proof: Put

$$g(t) = f(a + t(b-a))$$

$t \in [0, 1]$

Then $g: [0, 1] \rightarrow \mathbb{R}$

Then g is continuous □□□
on $[0, 1]$ and

by Chain rule Th 1.12,

g is differentiable

$$(g = f \circ h, \quad h(t) = a + t(b-a))$$

on $(0, 1)$ and

$$g'(t) = f'(h(t)) \cdot h'(t).$$

$$= f'(a + t(b-a)) (b-a)$$

— (1)

∴ By Mean Value Theorem from 17

elementary Calculus

∃ $t_0 \in (0, 1)$ st

$$g(1) - g(0) = g'(t_0)$$

$$f(b) - f(a) = f'(a + t_0(b-a)) \cdot (b-a)$$

Put $\xi = a + t_0(b-a) \in (a, b)$

Th 1.15 : (Mean Value

Theorem II) let f

$$f: [a, b] \subseteq \mathbb{R} \rightarrow Y$$

Y is a normed space.

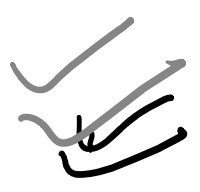
If for each $x \in (a, b)$

$f'(x)$ exists. If for

each $x \in (a, b)$, $f'(x)$

exists and satisfies

$$\|f'(x)\| \leq M$$



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then

$$\|f(b) - f(a)\| \leq M(b-a)$$

Illustration of this theorem

Eg. $f(x) = tx_0$ \ll
 y_0 is fixed.

Eg.

