Lecture 4

$f : \mathbb{R}^n \to \mathbb{R}^m$
and $x_0 \in \mathbb{R}^n$.

$f = (f_1, f_2, \ldots, f_m)$.

- $D_j f_i$ exists & is cts in a nbd of x_0
  $i = 1, 2, \ldots m$
  $j = 1, \ldots n$

Then f is Frechet Diff. at x_0

02

$$f'(x_0) \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^m)$$

$$\cong M_{m \times n}(\mathbb{R}).$$

$$f'(x_0) \cdot \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \begin{bmatrix} D_1 f_1 & \cdots & D_n f_1 \\ \vdots & & \vdots \\ D_1 f_m & \cdots & D_n f_m \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

$$= J h.$$

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

Jacobian

Th 1.10

L03.

Defn 1.11: Gateaux diff:

Let $D \subseteq X$, be open, X, Y normed spaces and $f : D \longrightarrow Y$. be given

f is stb Gateaux diff at $x_0 \in D$, iff $\exists$ a bdd linear map A s.t

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \| f(x+\lambda h) - f(x) - \lambda A h \| = 0$$

for every $h \in X$.

- If f is Frechet diff at x, then it is Gateaux diff at x and the two derivatives are equal.

- Converse ??
give examples

$\boxed{\text{Ex}}$

Theorem 1.1.2 : Chain Rule :

Let $f : D \longrightarrow Y$, $D \underset{\text{open}}{\subseteq} X$.

$g : E \longrightarrow Z$.

E open, $f(D) \subseteq E$.

If f is differentiable at x_0 and if g is diff at $f(x_0)$ then $g \circ f$ is diff at x_0 and,

$$\boxed{(g\circ f)'(x_0) = g'(f(x_0))\circ f'(x_0)}$$

<u>Pf</u>: Let $F = g\circ f$.

$A = f'(x_0) \in \mathcal{B}(X, Y)$

$B = g'(f(x_0))$

$\in \mathcal{B}(Y, Z)$.

We have to show that

$F'(x_0) = BA$.

$F(x_0+h) - F(x_0) - BA(h)$

$$= g(f(x_0+h)) - g(f(x_0)) - BA(h)$$

$$= g\big(f(x_0+h) - f(x_0) - Ah + f(x_0) + Ah\big) - g(f(x_0)) - BAh. \quad (1)$$

Let $o_1(h) = f(x_0+h) - f(x_0) - Ah$.

$$\frac{o_1(h)}{\|h\|} \longrightarrow 0 \text{ as } h \longrightarrow 0 \quad (2)$$

$$= g\big(\underbrace{f(x_0)}_{=y} + Ah + o_1(h)\big) - g(f(x_0)) - BAh.$$

$$= g(y + \psi(h)) - g(y) - BAh. \quad (3).$$

where $y = f(x_0)$
and. $\psi(h) = Ah + o_1(h)$.

L09.

$$g(y+\psi(h)) - g(y) + B\psi(h) - B\psi(h) - BAh.$$

$$= g(y) + B\psi(h) + \left(g(y+\psi(h)) - g(y) - B\psi(h)\right) - g(y) - BAh.$$

$$= g(y) + B\psi(h) + o_2(\psi(h)) - g(y) - BAh.$$

where

$o_2(\psi(h))$

$$= g(y+\psi(h)) - g(y) - B\psi(h).$$

&

$$\frac{o_2(\psi(h))}{\|\psi(h)\|} \to 0 \quad \text{if } \|\psi(h)\| \to 0$$

Mean Value Theorem:

Ex 1.13 $f : \mathbb{R} \longrightarrow \mathbb{R}^2$.

$g : \mathbb{R} \longrightarrow \mathbb{R}^2$.

$f(t) = (t, \sin t)$

$g(t) = (\cos t, \sin t)$.

Consider $f, g : [0, 2\pi] \longrightarrow \mathbb{R}^2$.

at any t, $f'(t), g'(t)$

are 2×1 matrices.

$$f'(t) = \begin{bmatrix} 1 \\ \cos t \end{bmatrix}$$

$$g'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

Does $\exists\, t_0 \in (0, 2\pi)$ st

$$f(2\pi) - f(0) = 2\pi f'(t_0)$$

$$g(2\pi) - g(0) = 2\pi g'(t_0).$$

$$(2\pi, 0)^T = 2\pi (1, \cos t_0)^T.$$

$t_0 = \pi/2$ works?

$$g(2\pi) - g(0)$$
$$= (1,0) - (1,0)$$
$$= (0,0).$$

$$\text{RHS} = 2\pi \begin{pmatrix} -\sin t_0 \\ \cos t_0 \end{pmatrix}$$

$\nexists$ any t_0. st.

$$\sin t_0 = 0 = \cos t_0.$$

Th 1.14.

Mean Value Theorem I

Let $f : D \longrightarrow \mathbb{R}$, where $D \subseteq X$ is open. Let $\boxed{a, b \in D}$. Assume that the line segment

$$[a,b] = \{a + t(b-a) : 0 \leq t \leq 1\}$$

lies in D.

If f is continuous on $[a,b]$ and diff on the open line segment (a,b), then for some $\xi \in (a,b)$

$$f(b) - f(a) = f'(\xi)(b-a).$$

$$\stackrel{?}{=} \cancel{(b-a)f'(\xi)}.$$

Proof: Put

$$g(t) = f(a + t(b-a)) \quad t \in [0,1].$$

Then $g : [0,1] \longrightarrow \mathbb{R}$

Then g is continuous on $[0,1]$ and by Chain rule Th 1.12, g is differentiable $\left(g = f \circ h, \quad h(t) = a + t(b-a) \right)$ on $(0,1)$ and

$$g'(t) = f'(h(t)) \cdot h'(t),$$

$$= f'(a+t(b-a))(b-a)$$

——— (1)

$\therefore$ By Mean Value Theorem from elementary Calculus

$\exists\, t_0 \in (0,1)$ st

$$g(1) - g(0) = g'(t_0).$$

$$f(b) - f(a) = f'(a + t_0(b-a))(b-a)$$

(using (1).)

Put $\xi = a + t_0(b-a) \in (a,b)$

Th 1.15 : (Mean Value Theorem II) Let f

$f : [a,b] \subseteq \boxed{\mathbb{R} \longrightarrow Y}$

Y is a normed space.

If for each $x \in (a,b)$

$f'(a)$ exists. If for each $x \in (a,b)$, $f'(x)$ exists and satisfies

$$\| f'(x) \| \leq M$$

then

$$\|f(b) - f(a)\| \leq M(b-a)$$

Illustration of this Thm

Eg: $f(t) = ty_0$.

y_0 is fixed.

Eg: