Chapter 3

Calculus in Banach Spaces

Theorem 1. If f is differentiable at x, th
definition is uniquely defined. (It depends on

Proof. Suppose that A_1 and A_2 are
erty, express

Topics in Analysis

Question : 1 prove that if f and g are differentiable at x then so $f+g$, and

$$(f+g)'(x) = f'(x) + g'(x)$$

Proof: Given, f and g both are differentiable at x_0, then

$$\lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - A_1 h \|}{\|h\|} = 0 \quad \text{--- (1)}$$

and

$$\lim_{h \to 0} \frac{\| g(x_0+h) - f(x_0) - A_2 h \|}{\|h\|} = 0 \quad \text{--- (2)}.$$

~~where $A_1 = f$~~

Now, consider

$$\lim_{h \to 0} \frac{\| (f+g)(x_0+h) - (f+g)(x_0) - A h \|}{\|h\|}$$

where ; $A = A_1 + A_2$.

$$\lim_{h \to 0} \frac{\| f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0) - A_1 h - A_2 h \|}{\|h\|}$$

$$= \lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - A_1 h + g(x_0+h) - g(x_0) - A_2 h \|}{\|h\|}$$

$$\leq \underbrace{\lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - A_1 h \|}{\|h\|}}_{\to 0} + \lim_{h \to 0} \frac{\| g(x_0+h) - f(x_0) - A_2 h \|}{\|h\|}$$

$$\Rightarrow \lim_{h \to 0} \frac{\| (f+g)(x_0+h) - (f+g)(x_0) - A h \|}{\|h\|} = 0 \quad \to 0$$

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$\therefore \quad (f+g)' = f' + g'.$

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Theorem 1. If f is differentiable at x, then the mapp... definition is uniquely defined. (It depends on x as well as...

Proof. Suppose that A_1 and A_2 are two linear maps having t... erty, expressed in Equation (1). Then to each $\epsilon > 0$ there ... such that

Question: Let g be a function of two real variable s.t. g_{22} is cts.

$f : C[0,1] \longrightarrow C[0,1]$

$$((f(x))(t) = \int_0^1 g(t, x(s))\, ds.$$

compute the frechet derivative of f. [Taylor series]..

Soln:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x+\theta h)$$

If f'' is cts at x then

$$f''(x+\theta h) \to f''(x) \text{ as } h \to 0$$

So, we can write $\quad \theta \in (0,1)$.

$$f''(x+\theta h) = f''(x) + \varepsilon(h) \text{ as } \varepsilon(h) \to 0$$

$$\frac{h^2}{2} f''(x+\theta h) = \frac{h^2}{2} f''(x) + o(h^2) \text{ as}$$

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Calculus in Banach Spaces

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3.2 The Chain Rule and M...
Newton's M...

Theorem 1. *If f is differentiable at x, then the mapping A in the definition is uniquely defined. (It depends on x as well as f.)*

Proof. Suppose that A_1 and A_2 are ... ng the required prop- erty, expressed in ... corresponds a $\delta > 0$

$\delta = \min\{\delta_1$

Now, we will write taylor series expansion of $g(t, x(s))$ w.r.t. 2nd argument. i.e.

$$g(t,(x+h)(s)) \simeq g_{20}(t,x(s)) + g_{21}(t,x(s))\frac{h}{1!}$$

$$\cancel{+ g_{22}(t,(x+\theta h))}$$

$$+ g_{22}(t, x+\theta h)\frac{h^2}{2!} \quad \text{---} \; (*)$$

given g_{22} is cts

$\Rightarrow$ 1st ~~consider~~ $Ah = \int_0^1 g_{21}(t,x(s))h\,dt$ — (1)

Consider,

$$\|f(x+h) - f(x) - Ah\|$$

$$= \left\| \int_0^1 g(t,(x+h)(s))\,dt - \int_0^1 g(t,x(s))\,dt - \int_0^1 g_{21}(t,x(s))h\,dt \right\|$$

$$= \left\| \int_0^1 \left[g(t,(x+h)(s)) - g(t,x(s)) - g_{21}(t,x(s))h \right] dt \right\|$$

$$= \left\| \int_0^1 g_{22}(t, x+\theta h)\frac{h^2}{2!} \right\|$$

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Now, $\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|}$

To show: $\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$

from (2)

$$\lim_{h \to 0} \frac{\left\| \int_0^1 g_{22}(t, x+\theta h) \frac{h^2}{2!} dt \right\|}{\|h\|}$$

$$\leq \lim_{h \to 0} \frac{\left\| \int_0^1 g_{22}(t, (x+\theta h)(s)) dt \right\| \|h^2\|}{2! \|h\|}$$

$$= \lim_{h \to 0} \frac{\left\| \int_0^1 g_{22}(t, x+\theta h)(s) dt \right\| \|h\|}{2!}$$

$\to 0$ as $h \to 0$

$$\Rightarrow \lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0$$

To show: A is bounded.

Consider $Ah = \int_0^1 g_{21}(t, x(s)) dt.$

$t \in [0,1]$.
$x(s)$ – map.
$C[0,1]$.

Given g_{22} is cts.

$\therefore$ g_{21} is also cts.

$\therefore$ g_{21} is bounded on closed interval $[0,1]$.

$\therefore$ Domain of g is $[0,1]$.

Notation. If f is differentiable at x, its derivative, denoted by A in the definition, will usually be denoted by $f'(x)$. Notice that with this notation $f'(x) \in \mathcal{L}(X, Y)$. This ... me as saying $f' \in \mathcal{L}(X, Y)$. It will be necess... f' and $f'(x)$.

...ighborhood of x and if a linear ...), then A is a bounded linear ...rivative of f at x.

... $\leq \delta$ we will have

... $- f(x) - Ah|| \leq ||h||$

... $2M + \delta$. For $||u|| \leq 1$, $||\delta u|| \leq \delta$,

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Hence, supremum will exist.

i.e. $||Ah|| = || \int_0^1 g_{21}(t, x(s)) h(s) ||$

$\leq || \int_0^1 ||g_{21}(t, x(s))|| \, ||h(s)|| \, ||h||$

finite. supremum.

$\therefore$ Integration of finite is finite.

$\Rightarrow ||Ah|| \leq K ||h||$

$K = \int_0^1 ||g_{21}(t, x(s))|| \, ds$

Hence A is bounded.