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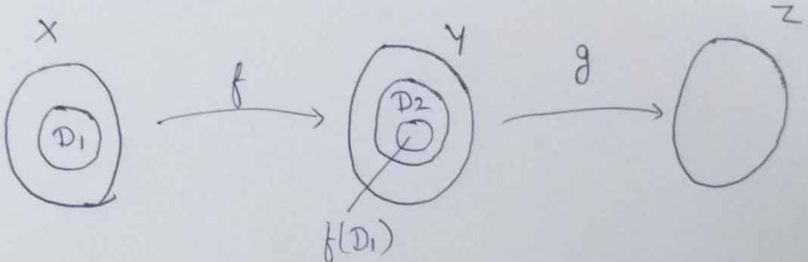
Theorem 1.9 The Chain Rule

Suppose X, Y & Z are three normed linear spaces. Let D_1 & D_2 be open subsets of X & Y resp. let $f : D_1 \longrightarrow Y$ & $g : D_2 \longrightarrow Z$ s.t. $f(D_1) \subset D_2$. Let f is differentiable at $x \in D_1$ & g be differentiable at $f(x) \in D_2$.

Then $g \circ f : D_1 \longrightarrow Z$ is differentiable at $x \in D_1$

and $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$

Proof:



define $F = g \circ f$

$A = f'(x)$, $y = f(x)$, $B = g'(y)$

Now, f is differentiable at $x \in D_1$, so by definition of Frechet derivative , $A = f'(x)$ is a bounded linear map

& $\lim\limits_{h \to 0} \dfrac{\| f(x+h) - f(x) - Ah \|_Y}{\|h\|_X} = 0$

Taking $r_1(h) = f(x+h) - f(x) - Ah$, we get ——— ①

$\dfrac{\| r_1(h) \|}{\|h\|} \longrightarrow 0$ as $h \longrightarrow 0$.

similarly , g is differentiable at $y = f(x) \in D_2$, so

take $r_2(k) = g(y+k) - g(y) - Bk$, ——— ②

then $\dfrac{\| r_2(k) \|}{\|h\|} \longrightarrow 0$ as $k \longrightarrow 0$.

Now , f is differentiable at $x \in D_1 \Rightarrow f$ is continuous at $x \in D_1$ (by theorem 1.3)

So, $f(x+h) = y + k$ for some $h \in X$ & $k \in Y$. ——— ③

Also $k \longrightarrow 0$ as $h \longrightarrow 0$ (by continuity of f at x)

Now, $k = f(x+h) - y$

$= f(x+h) - f(x)$

$= Ah + r_1(h)$

$\Rightarrow \|k\| \leq \|A\| \cdot \|h\| + \|r_1(h)\|$ ($\because$ A is a bounded L.O.)

$$\Rightarrow \frac{\|k\|}{\|h\|} \leq \|A\| + \frac{\|r_1(h)\|}{\|h\|} \quad \text{——— (4)}$$

Now, $F(x+h) - F(x) - BAh = g(f(x+h)) - g(f(x)) - BAh$

$= g(y+k) - g(y) - BAh$ (using ③)

$= Bk + r_2(k) - BAh$ (using ②)

$= B(k - Ah) + r_2(k)$ ($\because$ B is L.O.)

$= B(f(x+h) - f(x) - Ah) + r_2(k)$ (using ③)

$= B(r_1(h)) + r_2(k)$ (using ①)

Thus, $\dfrac{\|F(x+h) - F(x) - BAh\|}{\|h\|} = \|B(r_1(h)) + r_2(k)\| / \|h\|$

$\leq \|B\| \dfrac{\|r_1(h)\|}{\|h\|} + \dfrac{\|r_2(k)\|}{\|h\|}$ ($\because$ B is a bounded L.O.)

$= \|B\| \dfrac{\|r_1(h)\|}{\|h\|} + \dfrac{\|r_2(k)\|}{\|k\|} \cdot \dfrac{\|k\|}{\|h\|}$

$\leq \|B\| \dfrac{\|r_1(h)\|}{\|h\|} + \dfrac{\|r_2(k)\|}{\|k\|} \left(\|A\| + \dfrac{\|r_1(h)\|}{\|h\|} \right)$

(using ④)

$\longrightarrow 0$ as $h \longrightarrow 0$

Also, since A & B are bounded linear operators, so BA is also bounded linear operator. Thus, by the definition of Frechet derivative, $F = g \circ f$ is differentiable at $x \in D$, with $F'(x) = BA$.