Assignment

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Reference: Cheney (Q12, Problems 3.1, Page-120)

Question: Let $X = C[0,1]$, with its usual sup-norm. Select $t_i \in [0,1]$ and $v_i \in C[0,1]$, and define

$$f(x) = \sum_{i=1}^{n} [x(t_i)]^2 v_i.$$

Prove that f is differentiable at all points of X and give a formula for f'.
Solution:
Given:

$$f : C[0,1] \to C[0,1]$$
$$f(x) = \sum_{i=1}^{n} [x(t_i)]^2 v_i, \tag{1}$$

where $x, v_i \in C[0,1]$, and $t_i \in [0,1]$.
Now,

$$\begin{aligned} f(x+h) - f(x) &= \sum_{i=1}^{n} [(x+h)(t_i)]^2 v_i - \sum_{i=1}^{n} [x(t_i)]^2 v_i \\ &= \sum_{i=1}^{n} [x(t_i) + h(t_i)]^2 v_i - \sum_{i=1}^{n} [x(t_i)]^2 v_i \\ &= \sum_{i=1}^{n} [x(t_i)]^2 v_i + \sum_{i=1}^{n} [h(t_i)]^2 v_i + \sum_{i=1}^{n} [2x(t_i)h(t_i)] v_i - \sum_{i=1}^{n} [x(t_i)]^2 v_i \\ &= \sum_{i=1}^{n} [h(t_i)]^2 v_i + 2\sum_{i=1}^{n} [x(t_i)h(t_i)] v_i \end{aligned} \tag{2}$$

Thus, we can consider

$$Ah = 2\sum_{i=1}^{n} [x(t_i)h(t_i)] v_i, \tag{3}$$

or

$$A = 2\sum_{i=1}^{n}[x(t_i)]v_i$$

where $A : C[0,1] \to C[0,1]$ is a map that is to be claimed as bounded and linear map.
(i) For boundedness:

$$\begin{aligned} |Ah| &= |2\sum_{i=1}^{n}[x(t_i)h(t_i)]v_i| \\ &\leq 2\sum_{i=1}^{n}|x(t_i)h(t_i)|||v_i||_\infty \\ &\leq 2\sum_{i=1}^{n}||x||_\infty||h||_\infty||v_i||_\infty. \end{aligned} \tag{4}$$

So, A is bounded and

$$|A| \leq 2\sum_{i=1}^{n}||x||_\infty||v_i||_\infty.$$

(ii) For linearity:

Let us consider, two arbitrary functions $x, y \in C[0,1]$ and $p, q \in [0,1]$, then we need to show that

$$A(px + qy) = pA(x) + qA(y).$$

Now,

$$\begin{aligned} A(px+qy) &= \left(2\sum_{i=1}^{n}[x(t_i)]v_i\right)(px+qy) \\ &= \left(2\sum_{i=1}^{n}[x(t_i)]v_i\right)(px) + \left(2\sum_{i=1}^{n}[x(t_i)]v_i\right)(qy) \\ &= p\left(2\sum_{i=1}^{n}[x(t_i)]v_i\right)(x) + q\left(2\sum_{i=1}^{n}[x(t_i)]v_i\right)(y) \\ &= pA(x) + qA(y) \end{aligned} \tag{5}$$

Hence, A is a bounded and linear map from $C[0,1]$ to $C[0,1]$.
Thus, we have

$$f(x+h) - f(x) - Ah = \sum_{i=1}^{n}[h(t_i)]^2 v_i \tag{6}$$

So,

$$\begin{aligned}
lim_{h\to 0}\frac{||f(x+h)-f(x)-Ah||}{||h||} &= lim_{h\to 0}\frac{||\sum_{i=1}^{n}[h(t_i)]^2 v_i||}{||h||} \\
&\leq lim_{h\to 0}\frac{\sum_{i=1}^{n}|h^2(t_i)|||v_i||}{||h||} \\
&\leq lim_{h\to 0}\sum_{i=1}^{n}\frac{||h^2||||v_i||}{||h||} \\
&= lim_{h\to 0}\sum_{i=1}^{n}||h||||v_i|| \\
&\to 0
\end{aligned}$$

Hence, f is Fréchet differentiable at all points of $X = C[0,1]$ and

$$f' = A = 2\sum_{i=1}^{n}[x(t_i)]v_i$$