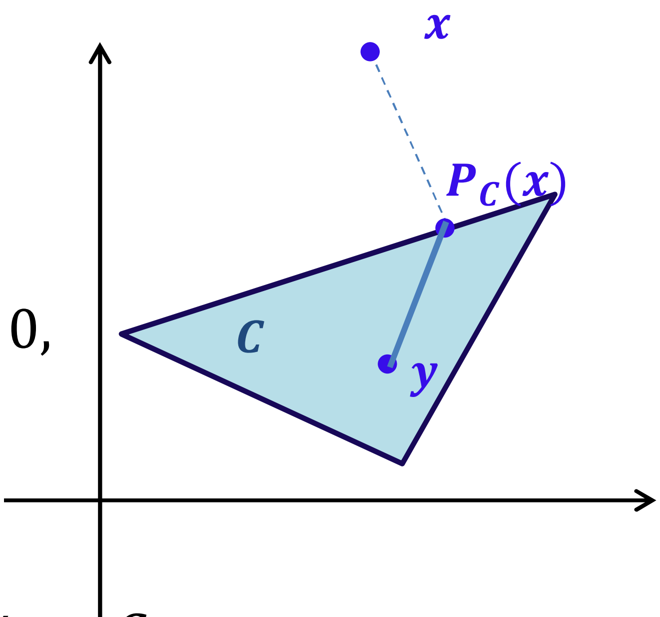
R-07
Convex and Nonsmooth Analysis

Separation of a convex Set and a Point

Theorem Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and let $x \notin C$. Then there exists $s \in \mathbb{R}^n$ such that

$$\langle s, x\rangle > \sup_{y\in C}\langle s, y\rangle.$$



Proof By projection inequality we have

$$\langle x - P_C(x), y - P_C(x)\rangle \le 0, \forall y \in C.$$

Since $x \notin C$ and C is closed $s := x - P_C(x) \neq 0$,

and hence we have

$$\langle s, y + s - x\rangle \le 0, \forall y \in C$$

which implies that

$$0 < \|s\|^2 \le \langle s, x - y\rangle, \forall y \in C.$$

Let $0 < t < \|s\|^2$.

$$\langle s, x\rangle > t + \langle s, y\rangle, \forall y \in C.$$

Then

$$\langle s, x\rangle \ge t + \sup_{y\in C}\langle s, y\rangle > \sup_{y\in C}\langle s, y\rangle.$$

Note We can choose s with $\|s\| = 1$.

Separation of a convex Set and a Point

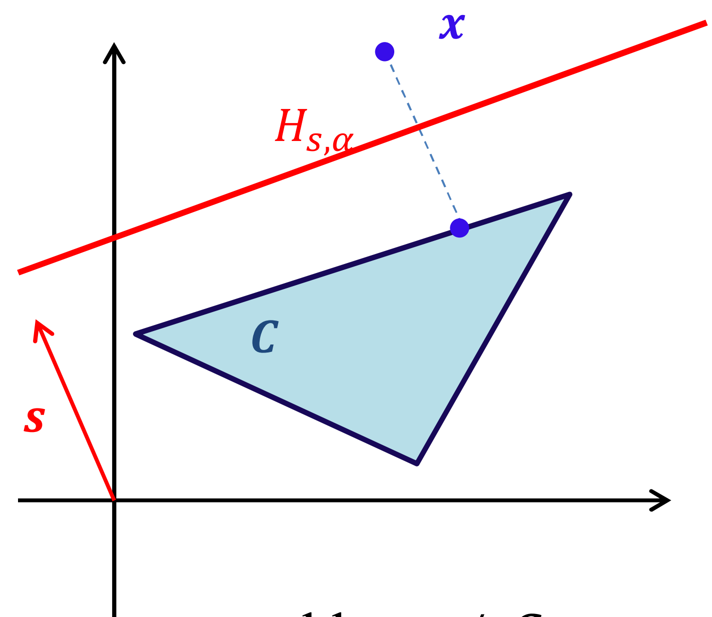
Corollary Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and let $x \notin C$. Then there exists $s \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$\langle s, x\rangle > \alpha > \langle s, y\rangle.$$

Proof Choose

$$\alpha = \frac{1}{2}(\langle s, x\rangle + \sup_{y \in C}\langle s, y\rangle).$$

The hyperplane $H_{s,\alpha}$ strictly separates C and $\{x\}$.



Theorem Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and let $x \notin C$. Then there exists $s' \in \mathbb{R}^n$ such that

$$\langle s', x\rangle < \inf_{y \in C}\langle s', y\rangle.$$

Proof Choose $s' = -s$ in previous theorem.

Support Function

Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. A function $\sigma_C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\sigma_C(s) := \sup_{y \in C} \langle s, y \rangle$$

is called support function of C.

i) $\sigma_C(ts) = t\sigma_C(s), \forall t > 0$, (positive homogeneous)

ii) $\sigma_C(s + s') = \sigma_C(s) + \sigma_C(s')$. (subadditive)

So, σ_C is a sublinear function. σ_C is a convex function

Example For $C = [-5, \infty[\subseteq \mathbb{R}$ we have $\sigma_C(s) = \begin{cases} -5s, & s \leq 0, \\ +\infty, & s > 0. \end{cases}$

Example For $C = [-1,2] \times [0,3] \subseteq \mathbb{R}^2$ we have

$$\sigma_C(s_1, s_2) = \begin{cases} 2s_1 + 3s_2, & s_1 \geq 0, s_2 \geq 0, \\ 2s_1, & s_1 \geq 0, s_2 < 0, \\ -s_1 + 3s_2, & s_1 < 0, s_2 \geq 0, \\ -s_1, & s_1 < 0, s_2 < 0 \end{cases}$$

$$= \max\{2s_1, -s_1\} + \max\{3s_2, 0\}.$$

Separation Theorem: Analytic form of Hahn–Banach Theorem

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and let $x \notin C$. Then there exists $s \in \mathbb{R}^n$ such that

$$\langle s, x \rangle > \sup_{y \in C} \langle s, y \rangle = \sigma_C(s).$$

Hence if $\sigma_C(s) \geq \langle s, x \rangle, \forall\, s \in \mathbb{R}^n$ then $x \in C$.

Also if $x \in C$ then $\sigma_C(s) = \sup_{y \in C} \langle s, y \rangle \geq \langle s, x \rangle, \forall\, s \in \mathbb{R}^n$.

$$x \in C \iff \sigma_C(s) \geq \langle s, x \rangle, \forall\, s \in \mathbb{R}^n.$$

Theorem (Hahn–Banach Theorem) Let W be a vector subspace of a vector space U. Let $f: W \to \mathbb{R}$ be linear and $p: U \to \mathbb{R}$ be sublinear. If for all $w \in W$, $f(w) \leq p(w)$, then there is a linear function $F: U \to \mathbb{R}$ such that

(i) $f(w) = F(w), \forall\, w \in W$,

(ii) $F(u) \leq p(u), \forall\, u \in U$.

Strict Separation of Two Convex Sets

Corollary Let C_1, C_2 be two nonempty closed convex sets in $\mathbb{R}^n$ with $C_1 \cap C_2 = \emptyset$. If C_2 is bounded, there exists $s \in \mathbb{R}^n$ such that

$$\min_{y \in C_2} \langle s, y \rangle > \sup_{y \in C_1} \langle s, y \rangle.$$

Proof The set $C_1 - C_2$ is convex and closed as C_2 is compact. As $C_1 \cap C_2 = \emptyset$ it follows that $0 \notin C_1 - C_2$. Hence by separation theorem there exists $s \in \mathbb{R}^n$ separating $\{0\}$ and $C_1 - C_2$, that is,

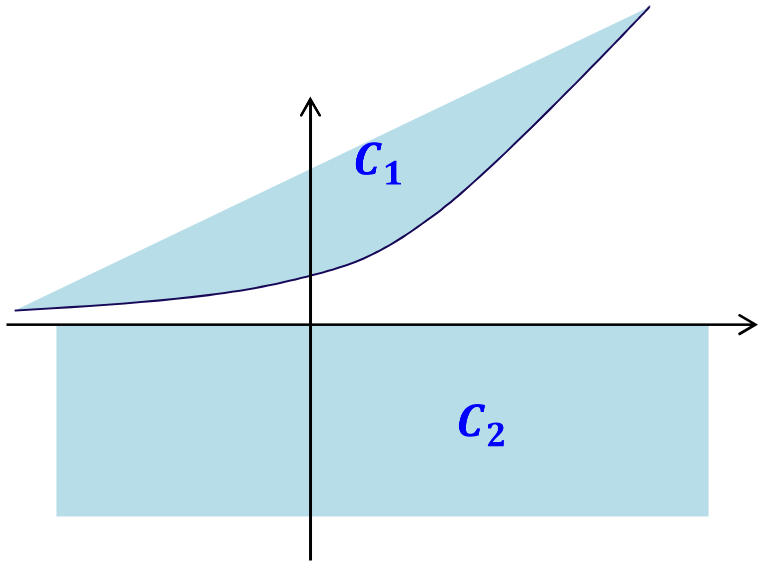
$$\begin{aligned} 0 = \langle s, 0 \rangle &> \sup_{y \in C_1 - C_2} \langle s, y \rangle \\ &= \sup_{y_1 \in C_1} \langle s, y_1 \rangle + \sup_{y_2 \in C_2} \langle s, -y_2 \rangle \\ &= \sup_{y_1 \in C_1} \langle s, y_1 \rangle - \inf_{y_2 \in C_2} \langle s, y_2 \rangle \\ &= \sup_{y_1 \in C_1} \langle s, y_1 \rangle - \min_{y_2 \in C_2} \langle s, y_2 \rangle \end{aligned}$$

as C_2 is compact.

Note $\sup_{y_1 \in C_1} \langle s, y_1 \rangle + \sup_{y_2 \in C_2} \langle s, -y_2 \rangle < 0 \Leftrightarrow \sigma_{C_1}(s) + \sigma_{C_2}(-s) < 0.$

Failure of Strict Separation

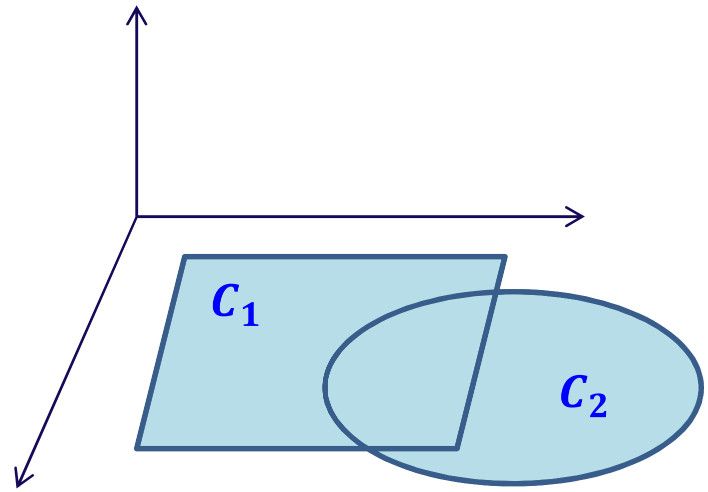
Let $C_1 = \{(x_1, x_2) : x_2 \geq e^{x_1}\}$ and $C_2 = \{(x_1, x_2) : x_2 \leq 0\}$.



Weak separation

Let C_1, C_2 be two nonempty closed convex sets in $\mathbb{R}^n$. They are weakly separated if

$$\inf_{y\in C_2}\langle s,y\rangle \geq \sup_{y\in C_1}\langle s,y\rangle.$$



$$\min_{y\in C_2}\langle(0,01),y\rangle = 0 = \max_{y\in C_1}\langle(0,0,1),y\rangle$$

The sets C_1 and C_2 are properly separated if

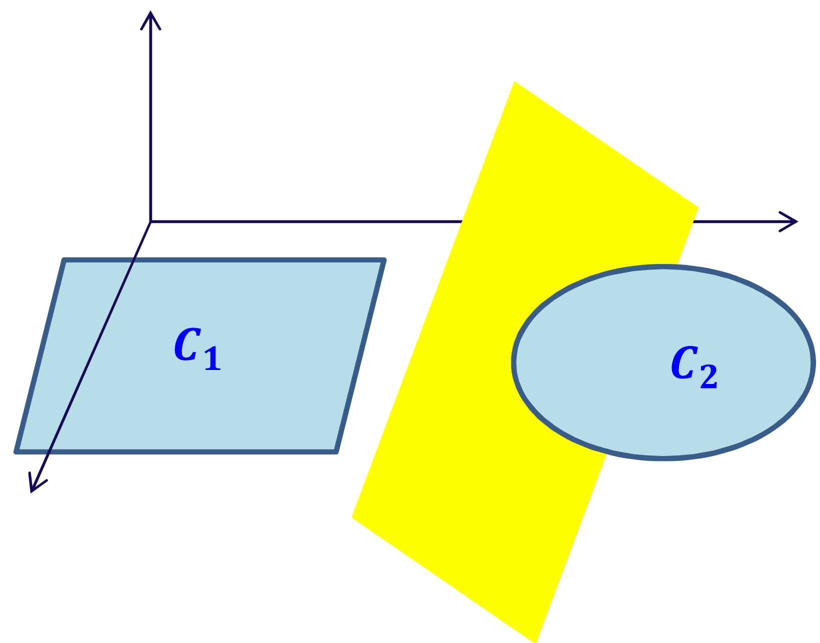
$$\inf_{y\in C_2}\langle s,y\rangle \geq \sup_{y\in C_1}\langle s,y\rangle \quad \& \quad \sup_{y\in C_2}\langle s,y\rangle > \inf_{y\in C_1}\langle s,y\rangle.$$

Theorem Two nonempty closed convex sets C_1 and C_2 in $\mathbb{R}^n$ can be properly separated if $\text{ri}C_1 \cap \text{ri}C_2 = \emptyset$.

Weak separation

If $s = (0,01)$ then

$$\min_{y \in C_2} \langle s, y \rangle = 0 = \sup_{y \in C_1} \langle s, y \rangle.$$



Separation is proper if $s = (1,0,0)$.

How to find a supporting hyperplane not containing whole of C?

Let C be a nonempty convex sets in $\mathbb{R}^n$ and $x \in \mathrm{bd}C$.

If $\mathrm{int}C = \emptyset$ either $x \in \mathrm{ri}C$ or $x \in \mathrm{rbd}C$.

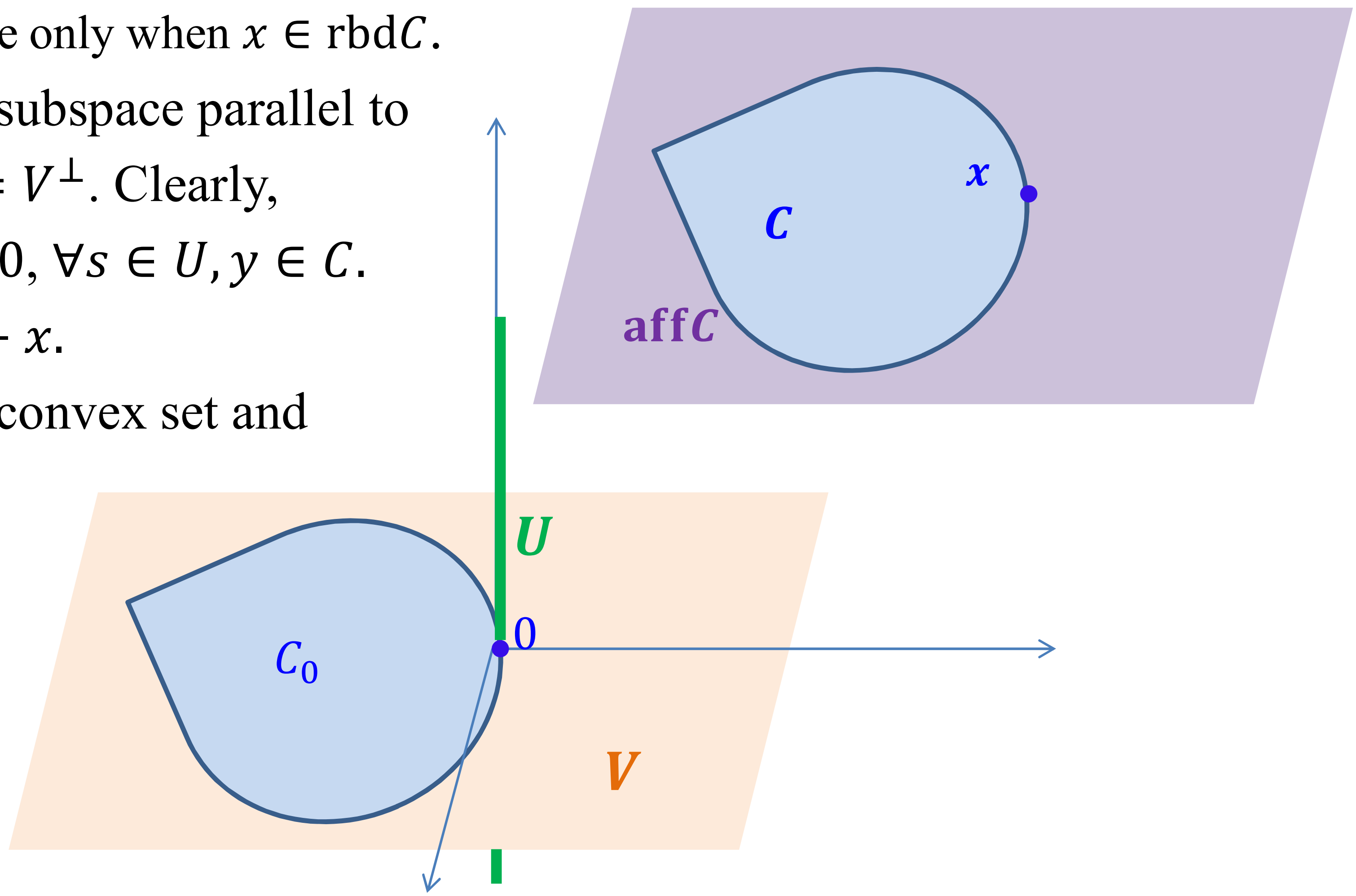
If $x \in \mathrm{ri}C$ then only supporting hyperplane is $\mathrm{aff}C$.

Supporting hyperplane not containing whole of C is possible only when $x \in \mathrm{rbd}C$.

Let V be the subspace parallel to $\mathrm{aff}C$ and $U = V^{\perp}$. Clearly, $\langle s, y - x\rangle = 0, \forall s \in U, y \in C$.

Let $C_0 = C - x$.

Then C_0 is a convex set and $0 \in \mathrm{rbd}C_0$.



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Let $\{x_k\} \subseteq V \backslash \mathrm{cl} C_0$ be such that $x_k \to 0$. Since $x_k \notin \mathrm{cl} C_0$ there exists $s_k \in V$ with $\|s_k\| = 1$ separating $\{0\}$ and C_0, that is,

$$\langle s_k, y \rangle \leq \langle s_k, 0 \rangle = 0, \forall y \in C_0.$$

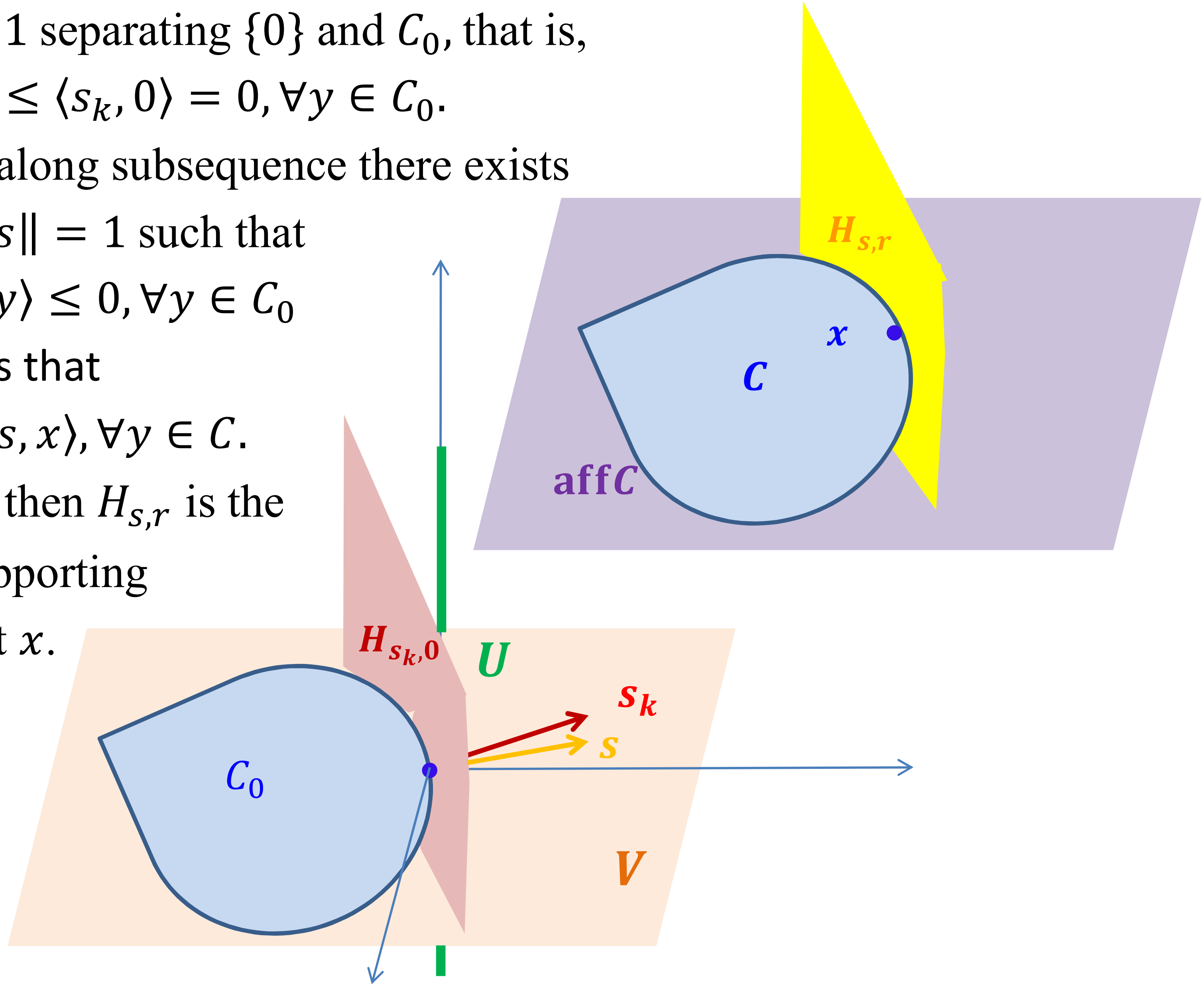
Taking limit along subsequence there exists $s \in V$ with $\|s\| = 1$ such that

$$\langle s, y \rangle \leq 0, \forall y \in C_0$$

which implies that

$$\langle s, y \rangle \leq \langle s, x \rangle, \forall y \in C.$$

If $r = \langle s, x \rangle$, then $H_{s,r}$ is the nontrivial supporting hyperplane at x.



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In fact $H_{s,r}$ is the nontrivial supporting hyperplane at x if $s \notin U$.

Any s with the decomposition $s = s_V + s_U$ is a nontrivial supporting hyperplane at x if $s_V \neq 0$. Otherwise, for all $y \in C$ we have

$$r = \langle s, y \rangle = \langle s_V, y \rangle + \langle s_U, y \rangle = \langle s_V, y \rangle + \langle s_U, y - x \rangle + \langle s_U, x \rangle$$
$$= \langle s_V, y \rangle + \langle s_U, x \rangle$$

which implies that $\langle s_V, . \rangle$ is constant on C.

This further implies that $\langle s_V, . \rangle$ is constant on aff$C$ and hence $s_V \in U$.

As $U \cap V = \{0\}$ it follows that $s_V = 0$ which is a contradiction.