Advanced Fluid Dynamics.

Boundary layer theory (MATH14 - 404(B))

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Reference Book.

Boundary layer Theory by Hermann Schlichting

- Part A, chapter I
  (a)/5, b/6, e/12,
  Chapt - II
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- Part-B. chapter - VII
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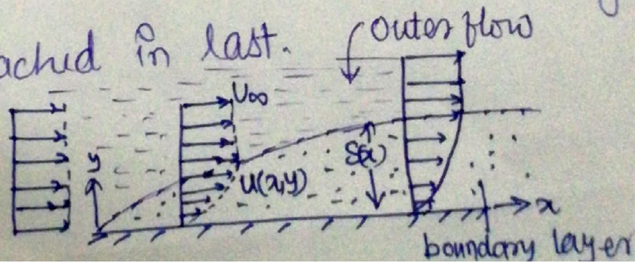
Concept (Boundary layer)

Boundary layer theory deals with the motion of viscous fluid in contact of a surface of a body (or wall). The influence of viscosity at high Reynolds number is confined to a very thin layer in the immediate neighbourhood of the solid wall. In thin layer the velocity of the fluid increases from zero at wall (no slip condition) to its full value which corresponds to external non-viscous flow. The layer under consideration is called boundary layer. The concept & theory of boundary layer was proposed by Ludwig Prandtl in 1904 in his historic paper.

In semester II, we have learned about Magnus effect, which predicts about lift on a body immersed in a non-viscous fluid in presence of circulation about it, but the theory say nothing about origin of circulation. At the same time, the drag on immersed bodies could not be explained by potential theory (ie theory of inviscid flows).

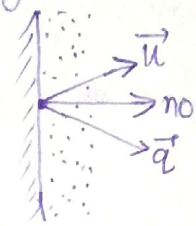
Ludwig Prandtl proposed that the viscous flow for high Reynolds number in contact of a wall can be divided into two regions: the inviscid outer flows and the boundary layer flow (very thin layer in contact of wall). In boundary layer, the Navier-Stokes's equation are simplified significantly to apply. The two flows are matached in last.



No slip condition

In semester two, we have studied the boundary condition as zero relative velocity of the fluid along the normal to the wall.

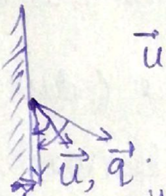
In viscous fluid flows, the no slip condition is defined as zero relative velocity of the fluid ~~along the~~ at the wall in tangential direction of the wall.



$\vec{n}$ — unit normal $\qquad \vec{n}\cdot\vec{u} = \vec{n}\cdot\vec{q}$

Boundary condition.

$\vec{u}\cdot\vec{t} = \vec{q}\cdot\vec{t}$



$\vec{t} \rightarrow$ unit tangent vector

Example: The sticking of dust on the blade of ceiling fan is an example of no-slip condition.

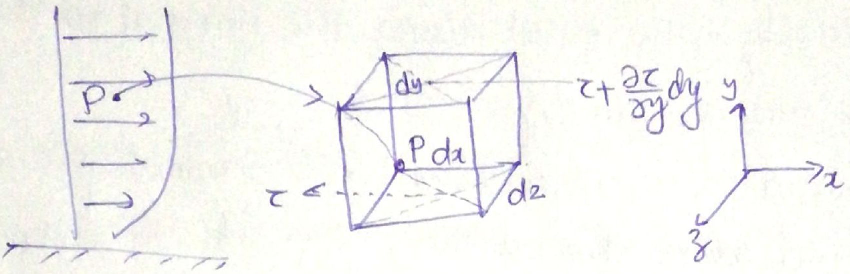
The principle of similarity: the Reynolds number and Mach Numbers

→ The two motions, which have geometrically streamlines, are called dynamically similar, or similar flows.

→ For two flows about geometrically similar bodies (e.g. two spheres), with different fluids, different velocities and different linear dimensions are similar flows if at all geometrically similar points, the forces acting on fluid particles must bear a fixed ratio at every instant of time.

→ Now in viscous incompressible fluid flow, we find the ratio of frictional (viscous) force to inertial forces.

Consider, a viscous fluid element in shape of parallelopiped of dimension dx, dy and dz at a point P(x, y, z)



Let, the fluid element is moving with velocity u in x-direction. Then the ~~initial force (acting due to~~ inertial force (due to momentum in the fluid element) acting on fluid element is equal to $\rho \frac{du}{dt} dx dy dz = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) dx dy dz$. For steady flow, the inertial force per unit volume $= \rho u \frac{\partial u}{\partial x}$. ($\rho$ = density of fluid)

By Newton's law of viscosity, the resultant of shearing stress forces acting on fluid element in x-direction is

$$= \left(\tau + \frac{\partial \tau}{\partial y} dy\right) dx dz - \tau dx dz$$

$$= \frac{\partial \tau}{\partial y} dx dy dz$$

$$= \mu \frac{\partial^2 u}{\partial y^2} dx dy dz \qquad \left(\because \tau = \mu \frac{\partial u}{\partial y}\right)$$

where μ is coefficient of dynamic viscosity.

hence, the friction force per unit volume of the fluid is

$$= \mu \frac{\partial^2 u}{\partial y^2}.$$

Therefore, the condition of similarity is written as

$$\frac{\text{Inertial force}}{\text{Friction force}} = \frac{\rho u \frac{\partial u}{\partial x}}{\mu \frac{\partial^2 u}{\partial y^2}} = \text{Constant.}$$

Consider, the fluid (viscosity μ, density ρ) flow has free-stream velocity **V**, characteristic ~~stream~~ linear dimension d (diameter of spheres) then,

$$\frac{\text{inertial force}}{\text{friction force}} = \frac{\rho V \frac{V}{d}}{\mu \frac{V}{d^2}} \quad \text{(from dimensional analysis)}$$

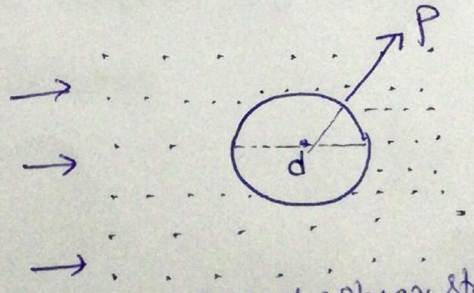
$$= \frac{\rho V d}{\mu}$$

Condition of similarity is satisfied if quantity $\frac{\rho V d}{\mu}$ has same values in both flows. The quantity $\frac{\rho V d}{\mu}$ is a dimensionless number (ratio of two forces) known as Reynolds number (Re).

$$Re = \frac{\rho V d}{\mu} = \frac{Vd}{\mu/\rho} = \frac{Vd}{\nu}, \quad \nu = \frac{\mu}{\rho} = \text{kinematic viscosity}$$

Drag and lift–

Consider, an immersed body in a fluid stream of viscous fluid



normal & shear stress act as surface force

let P is component of resultant force in any direction, then

$$\frac{P}{\rho V^2 d^2} = \text{dimensionless force}$$

or

$$\frac{P}{\frac{1}{2}\rho V^2 A} = \text{dimensionless force}$$

where, A: frontal area exposed by the body to the flow direction.

In case of a flow over a body or motion of a body in a fluid, two types of important forces are acting namely drag and lift, which have wide applications.

The component of resultant force parallel to the undisturbed initial velocity is called drag D, and the component perpendicular to that direction is called lift L. Therefore, ~~components of drag~~ the dimensionless coefficients for lift and drag is defined as

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 A} \quad \text{and} \quad C_D = \frac{D}{\frac{1}{2}\rho V^2 A}$$

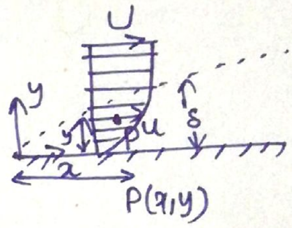
For, geometrically similar system, the lift and drag coefficients C_L and C_D are function of one variable only ie Reynold number R_e.

ie. $C_L = f_1(R_e)$, $C_D = f_2(R_e)$.

This is only true for incompressible fluid flow in absence of gravitational force. For compressible fluid, we have to take in consideration the Mach number $M = \frac{V}{a}$ and for free stream flow, the Froude number $F = V/\sqrt{gd}$, where a is speed of sound and g is acceleration due to gravity.

Estimation of boundary layer thickness (page26).

In boundary layer, the viscous (friction) force is significant for high Reynolds number flow and is of comparable order as inertia force. In case of laminar steady flow over a ~~flat~~ plate of length l, the initia force per unit volume is $\rho u \frac{\partial u}{\partial x}$ and friction force per unit volume is $\frac{\partial \tau}{\partial y}$ as discussed earlier. Therefore,



friction force $\sim$ inertia force

$$\frac{\partial \tau}{\partial y} \sim \rho u \frac{\partial u}{\partial x}$$

or, $$\mu \frac{\partial^2 u}{\partial y^2} \sim \rho u \frac{\partial u}{\partial x}$$ (for laminar flow $\tau = \mu \frac{\partial u}{\partial y}$)

dimensionally or, $$\mu \frac{U}{\delta^2} \sim \rho U \cdot \frac{U}{l}$$

or, $$\delta \sim \sqrt{\frac{\mu l}{\rho U}} = \sqrt{\frac{\nu l}{U}}$$

For lamilar flow, from the exact sol^n of boundary layer given by H. Blasius

$$\delta = 5\sqrt{\frac{\nu l}{U}}$$

The dimensionless bd layer thickness, referred to the plate length l becomes

$$\frac{\delta}{l} = 5\sqrt{\frac{\nu}{Ul}} = \frac{5}{\sqrt{R_l}}$$, R_l = Reynolds no.

number related to the length of the plate l, $R_l = \frac{\nu}{Ul}$

Boundary layer thickness at leading edge distance x, can also be defined as $$\delta = 5\sqrt{\frac{\nu x}{U}}$$ #

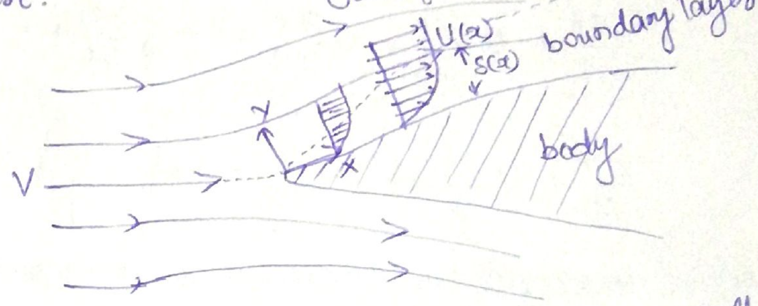
and as $R \to \infty$, $\delta/l \to 0$.

Question 1. Compute the shearing stress τ_0 on the wall of length l and breath b for a fluid flow of density ρ, viscosity μ and uniform stream speed U. (Page 27)

2. Compute the total drag D on the plate in question 1 and estimate the drag coefficient C_D. (Page 27).

Boundary layer equations for two-dimensional flow:

Consider, two dimensional laminar flow of incompressible fluid with small viscosity (high Reynolds number flow) about a cylindrical body of slender cross-section as shown in figure.



boundary layer flow along a wall

The flow over body can be divided into a outer flow and boundary layer flow. Let the density of the fluid is ρ and at a point $P(x,y)$ in boundary layer region the velocity is $\vec{q} = u\hat{i} + v\hat{j}$.

This flow can be seen as

1. the velocity of the fluid at wall is zero (no slip condition). In boundary layer the velocity gradient $\frac{\partial u}{\partial y}$, normal to the wall is very large. Therefore, the shearing stress $\tau = \mu \frac{\partial u}{\partial y}$ is very large even for very small viscosity of the fluid.

2. In outer flow, there is no such large velocity gradient therefore, for small viscosity of the fluid, the shearing stress is negligible and so outer flow can be taken as frictionless and potential kind.

In outer flow, the Euler's equation of motion can be used and in boundary layer, the Navier Stokes's equation of motion will be used that can be simplified to great extent due to the boundary layer properties and low viscosity of the fluid.

The equation of continuity, equation of motion (NS equation) and boundary conditions for ~~two~~ the two dimensional flow in boundary layer are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{— (1)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{— (2)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad \text{— (3)}$$

where external body forces are neglected.

BC's. no slip condition at wall: $u=0,\ v=0$ at $y=0$. — (4)

$u = U(x)$ as $y \to \infty$.

Now, to simplify above eq^n we first reduce eq^n (1)-(4) into non-dimensional form using transformation

$$x' = \frac{x}{L},\quad y' = \frac{y}{L},\quad t' = \frac{t}{L/V},\quad p' = \frac{p}{\rho V^2},\quad u' = \frac{u}{V},\quad v' = \frac{v}{V} \text{ and}$$

using Reynolds number $Re = \frac{VL\rho}{\mu} = \frac{VL}{\nu}$, V is uniform stream velocity (char. velocity) and L is char. length, then (1)-(4) reduces into

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

$$\frac{V\,\partial u'}{L/V\,\partial t'} + \frac{u'V^2}{L}\frac{\partial u'}{\partial x'} + \frac{v'V.V}{L}\frac{\partial v'}{\partial y'} = -\frac{1}{\rho}\frac{\rho V^2}{L}\frac{\partial p'}{\partial x'} + \frac{\nu.V}{L^2}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right)$$

$\div \frac{V^2}{L}$,

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \underbrace{\frac{\nu}{VL}}_{\frac{1}{Re}}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right)$$

Similarity (3) can be written as

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\frac{\partial p'}{\partial y'} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right)$$

and $u'=0,\ v'=0$ at $y'=0$ and $u' = \frac{U}{V} = U'$ as $y \to \infty$

replacing $x' \to x,\ y' \to y,\ t' \to t,\ u' \to u,\ v' \to v,\ p' \to p$

in above equations we get

$$\underset{1}{\frac{\partial u}{\partial x}} + \underset{1}{\frac{\partial v}{\partial y}} = 0 \quad -(5)$$

$$\underset{1}{\frac{\partial u}{\partial t}} + \underset{1\ \ 1}{u\frac{\partial u}{\partial x}} + \underset{\delta \cdot \frac{1}{\delta}}{v\frac{\partial u}{\partial y}} = -\frac{\partial p}{\partial x} + \underset{\delta^2}{\frac{1}{Re}}\left(\underset{1}{\frac{\partial^2 u}{\partial x^2}} + \underset{\frac{1}{\delta^2}}{\frac{\partial^2 u}{\partial y^2}}\right) \quad -(6)$$

$$\underset{\delta}{\frac{\partial v}{\partial t}} + \underset{1\ \ \delta}{u\frac{\partial v}{\partial x}} + \underset{\delta\ \ 1}{v\frac{\partial v}{\partial y}} = -\frac{\partial p}{\partial y} + \underset{\frac{1}{\delta^2}}{\frac{1}{Re}}\left(\underset{\delta}{\frac{\partial^2 v}{\partial x^2}} + \underset{\frac{1}{\delta}}{\frac{\partial^2 v}{\partial y^2}}\right) \quad -(7)$$

and $u=0,\ v=0$ at $y=0$ and $u=U$ as $y\to\infty$. (8).

Now, we apply the order (or scale) analysis for simplification of (5)-(8) in boundary layer. We know that, for laminar flow, the non-dimensional boundary layer thickness $\frac{\delta}{L} \sim \sqrt{\frac{\nu}{VL}} \sim \sqrt{\frac{1}{Re}}$, therefore, the nondimensional thickness $\delta' = \frac{\delta}{L} \sim \sqrt{\nu}$ and $\frac{1}{\sqrt{Re}}$.

Further, in boundary layer $\frac{\delta}{L} << 1$ ($\because Re \to \infty$ and $\nu \to 0$)

therefore, replacing $\delta' = \frac{\delta}{L}$ by $\delta << 1$ — (9).

Now, we compute the order of every terms in eq^n (5) to (8) in bd layer. Now, every term of finite order and small order will be denoted by symbols $O(1)$ and $O(\delta)$ respectively ($\because \delta << 1$).

For possible motion in bd layer, eq^n of continuity (5) must holds and $\frac{\partial u}{\partial x}$ must be finite ie of order $O(1)$, therefore $\frac{\partial v}{\partial y} = O(1)$ and $v=0$ at $y=0$ and $y=O(\delta)$ in bd layer, therefore $v=O(\delta)$.

Also, in boundary layer $x=O(1)$ and $u=O(1)$, therefore,

$\frac{\partial v}{\partial x}$ and $\frac{\partial^2 v}{\partial x^2}$ both are $O(\delta)$ and $\frac{\partial^2 u}{\partial x^2} = O(1)$.

Further, we assume that the non-steady acceleration $\frac{\partial u}{\partial t} = O\left(u\frac{\partial u}{\partial x}\right) = O(1)$ ie very sudden acceleration are excluded.

Since, in boundary layer, the inertia term are of same order as viscous term and $\sqrt{\frac{1}{Re}} = O(\delta)$, therefore $\frac{1}{Re} = O(\delta^2)$, therefore, $\frac{\partial^2 u}{\partial y^2} = O\left(\frac{1}{\delta^2}\right)$ and $\frac{\partial^2 v}{\partial y^2} = O\left(\frac{\delta}{\delta^2}\right) = O\left(\frac{1}{\delta}\right)$ (since $u=0$ at $y=0$ and $u = O(1)$ as $y \to \infty$, therefore, $\frac{\partial u}{\partial y} = O(\delta)$). Now we can put the order or magnitude of each term in eq^{n}s (5), (6) and (7) (see it)

The eq^n (5) remains unchanged for very large Reynolds number.

The eq^n (6) is simplified by neglecting $\frac{\partial^2 u}{\partial x^2}$ w.r.to $\frac{\partial^2 u}{\partial y^2}$. In eq^n (8), $\frac{\partial p}{\partial y} = O(\delta)$, because all other term are of order $O(\delta)$, therefore $p = O(\delta^2)$ ie pressure variation in direction normal to the boundary layer is very small and so pressure is constant in boundary layer and may be assumed equal to pressure at outer edge of boundary layer that can be calculated from the frictionless flow in outer region. and therefore, pressure can be considered as known in boundary layer and is a function of x & t only i.e. $p = p(x,t)$. The equation (8) is dominated by equation (7) in magnitude for boundary layer flow.

At the outer edge of boundary layer, $u = U(x,t)$ and $\tau = \mu\frac{\partial u}{\partial y} \sim 0$ ($\because$ μ is small & $\frac{\partial u}{\partial y} \to 0$), therefore, for outer flow,

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{(dimensional form)}. \qquad (10)$$

For steady flow,

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx}$$

on integration, $p + \frac{1}{2}\rho U^2 = \text{constant}$ — (11)

Now, using all the above analysis (scale), the simplified Navier-Stokes's equation of motion for boundary layer is known as Prandtl's boundary layer equations and can be written in ~~non~~ dimensional form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{— (12)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{— (13)}$$

with BC's at $y=0$ (wall), $u=0$, $v=0$; and as $y \to \infty$ $u = U(x,t)$. The potential flow $U(x,t)$ is considered to be known and it determines the pressure from eq^n (10) or (11). In addition, the initial condition (at $t=0$) must be prescribed over the whole xy region.

In case of steady flow, the Prandtl B.L. eq^{ns} are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

with boundary conditions

$y=0$: $u=0$, $v=0$;

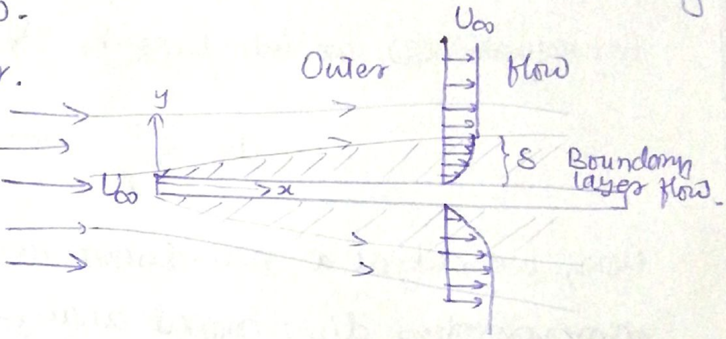
$y \to \infty$; $u = U(x)$.

It is also necessary to prescribe velocity profile at initial section $x = x_0$ i.e. $u(x_0, y)$.

Boundary layer along a flat plate :-
(page 135)

(This is application of Prandtl boundary layer theory for a flow along a very thin flat plate)

Let the consider steady motion of incompressible laminar flow with free stream velocity U_∞ over a flat plate being parallel to x-axis and infinitely long with leading edge at $x=0$.
The fluid is of high Reynolds number.



For outer flow:

$$U_\infty \frac{dU_\infty}{dx} + \frac{1}{\rho}\frac{dp}{dx} = 0 \quad \text{(steady flow)}$$

$$\Rightarrow \quad 0 + \frac{1}{\rho}\frac{dp}{dx} = 0$$

$$\Rightarrow \quad p(x) = \text{Constant.} \qquad \text{— (1)}$$

The boundary layer flow. (steady)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad \text{— (2)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{=0} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{— (3)}$$

BC's. $y=0$: $u=0$, $v=0$; and at $y=\infty$: $u=U_\infty$ — (4).

Since, the plate is infinitely long i.e the system has no preferred length therefore, the velocity curves $u(y)$ with x can be made identical by selecting suitable scale factor for u and y. The scale factor for u and y are free stream velocity U_∞ and boundary layer thickness $\delta(x)$, respectively. Therefore, principle of similarity for velocity profile in boundary layer can be written as $u = U_\infty \phi(y/\delta)$, where ϕ must be same at all distances x from the leading edge.

From the exact solution of boundary layer flow over a suddenly accelerated plate, $\delta \sim \sqrt{\nu t}$, t being time from the start of the motion. For a particle outside the boundary layer $t = \frac{x}{U_\infty}$, therefore

$$\delta \sim \sqrt{\frac{\nu x}{U_\infty}}$$

Now, define a dimensionless similarity variable $\eta \sim \frac{y}{\delta}$ so that

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \quad \text{—(5)}$$

For equation (2), we introduce the stream function $\psi(x,y)$ given by

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \quad (6)$$

Now, we define a non-dimensional stream function $f(\eta)$ from the ~~dimensionless~~ dimensional analysis as

$$\psi(x,y) = \sqrt{\nu x U_\infty}\, f(\eta). \quad \text{—(7)}$$

$[\psi] = L^2T^{-1}$

and

$$[\nu x U_\infty] = \left[\frac{\mu}{\rho} \cdot x \cdot U_\infty\right]$$

$$= [\mu][x][U_\infty]/[\rho]$$

$$= \left[\frac{\tau}{\frac{\partial u}{\partial y}}\right] \cdot [x][U_\infty]/[\rho]$$

$$= \frac{ML^{-1}T^{-2}}{\frac{LT^{-1}}{L}} \cdot L \cdot LT^{-1} \cdot \frac{L^3}{M}$$

$$= L^4T^{-2}$$

$$\therefore \left[\sqrt{\nu x U_\infty}\right] = L^2T^{-1}$$

Therefore u and v can be obtained as.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[\sqrt{\nu x U_\infty}\, f\right] = \sqrt{\nu x U_\infty}\,\frac{\partial f}{\partial y}$$

$$= \frac{\partial}{\partial \eta}[f]\,\frac{\partial \eta}{\partial y} \cdot \sqrt{\nu x U_\infty}$$

$$= \sqrt{\nu x U_\infty} \cdot f'(\eta) \cdot \sqrt{\frac{U_\infty}{\nu x}} \quad \text{(using (5))}$$

$$u = \frac{\partial \psi}{\partial y} = U_\infty f'(\eta).$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[\sqrt{\nu x U_\infty}\, f(\eta)\right]$$

$$= -\left[\sqrt{\nu U_\infty}\,\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\, f(\eta) + \sqrt{\nu x U_\infty}\, f'(\eta)\frac{\partial \eta}{\partial x}\right]$$

$$= -\left[\frac{\sqrt{\nu U_\infty}}{2\sqrt{x}}\, f(\eta) + \sqrt{\nu x U_\infty} \cdot f'(\eta) \cdot y \cdot \sqrt{\frac{U_\infty}{\nu}} \cdot -\frac{1}{2}\frac{1}{x\sqrt{x}}\right]$$

$$= -\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[f(\eta) - \eta f'\right]$$

Similarly, we compute $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ needed in eq^n ③. 16

as
$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{U_\infty}{x}\eta f''(\eta)$$
$$\frac{\partial u}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}\, U_\infty f''$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} f'''$$

Therefore, using all in eq^n ③, we get $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = +\nu\frac{\partial^2 u}{\partial y^2}$

so.
$$\cancel{U_\infty} f'(\eta)\cdot -\frac{1}{2}\frac{U_\infty}{x}\eta f''(\eta) + \left(-\frac{1}{2}\right)\sqrt{\frac{\nu U_\infty}{x}}\left[f - \eta f'\right]\cdot\sqrt{\frac{U_\infty}{\nu x}}\, U_\infty f''$$
$$= \cancel{\nu}\,\frac{U_\infty^2}{\cancel{\nu} x} f'''(\eta)$$

or,
$$-\eta f' f'' - (f - \eta f')\cdot f'' = 2f'''$$
$$-\cancel{\eta f' f''} - f f'' + \cancel{\eta f' f''} = 2f'''$$

or
$$\boxed{f\cdot f'' + 2f''' = 0.}$$ Blasius's Equation

and reduced BC's.

when $\boxed{\eta = 0: \quad f = 0,\ f' = 0.}$ and as $\boxed{\eta \to \infty, \quad f' = 1.}$

Thus, the ~~Prad~~ Prandtl boundary layer equations reduces into a third order non-linear boundary value problem. Different people has tried to solve the Blasius equation under bd conditions by different method over period of time. Solution is ploted on page 137 and tabulated on page 139 of the book.

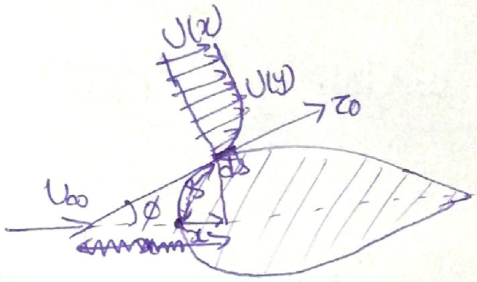
Skin Friction (viscous drag)

The viscous drag or skin friction around a surface can be obtained by integrating the shearing stress at the wall.

The shearing stress at the wall is $\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0 \text{ (wall)}}$

The viscous drag for two dimensional flow over a cylindrical body of height b is

$$D_f = b \int_{s=0}^{l} \tau_0 \cos\phi \, ds$$



Integration is to be performed over whole wetted surface, from leading to trailing edge assuming no seperation.

- Using $ds \cos\phi = dx$, x is measured parallel to the free stream velocity, therefore

$$D_f = b\mu \int_{s=0}^{l} \left(\frac{\partial u}{\partial y}\right)_0 dx.$$

For flat plate case, where length of the plate is l,

$$\tau_0 = \left(\frac{\partial u}{\partial y}\right)_{y=0} \cdot \mu = \mu \cdot U_\infty \sqrt{\frac{U_\infty}{\nu x}} f''(0) = \alpha \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}}.$$

where $\alpha = f''(0) = 0.332$ from numerical sol^n of Blassius equation.

The dimensionless shearing stress or drag coefficient C_f is

$$\frac{1}{2} C_f = \frac{\tau_0(x)}{\rho U_\infty^2} = \frac{0.332 \, U_\infty \mu \sqrt{\frac{U_\infty}{\nu x}}}{\rho U_\infty^2} = 0.332 \sqrt{\frac{\nu}{U_\infty x}} = \frac{0.332}{\sqrt{Re_x}}$$

and Skin friction $D_f = \alpha \mu b U_\infty \sqrt{\frac{U_\infty}{\nu x}} \int_0^l \frac{dx}{\sqrt{x}} = 2\alpha b U_\infty \sqrt{\mu \rho l U_\infty}$

and for wetted plate on both side $2D = 1.328 \, b \sqrt{U_\infty^3 \mu \rho l}$ #

Boundary layer thickness:

There are three types of measurement of boundary layer thickness:

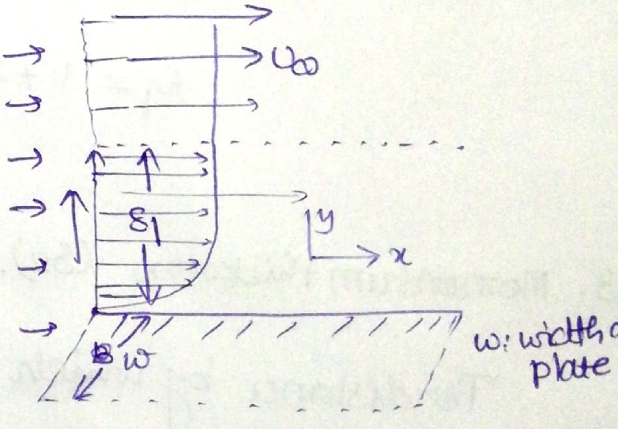
1. disturbance thickness
2. displacement thickness or mass thickness
3. momentum thickness.

1. disturbance thickness: (δ)

The disturbance thickness is defined as distance from the wall in y-direction (or normal direction) to a point where parallel component u of the velocity in boundary layer is equal to $0.99U_\infty$ ie. $u = 0.99U_\infty$. From the numerical solution of boundary layer over a flat plate, the disturbance thickness δ is given by

$$\delta \cong 5.0\sqrt{\frac{\nu x}{U_\infty}}$$

2. Mass or displacement thickness: (δ_1)



The displacement thickness (δ_1) is the distance by which plate must move in normal direction (to the plate) so that loss of mass flux is equal to the loss of mass flux due to boundary layer. The shifting of plat causes the loss of mass flux due to the decrease in the cross-section area of the flow.

loss of mass flux due to plate ~~motion~~ movement $= \int_0^{\delta_1} U_\infty \rho w\,dy$

$= \rho w U_\infty \delta_1$

loss of mass flux due to boundary layer $= \int_0^{\infty} U_\infty \rho w\,dy - \int_0^{\infty} u\rho w\,dy$

$= \int_0^{\infty} (U_\infty - u)\, w\rho\,dy$

by definition of mass/displacement thickness

$$\rho \cancel{w} U_\infty \delta_1 = \cancel{w} \cdot \int_0^\infty \rho (U_\infty - u)\, dy$$

In case of incompressible fluid (ρ = Constant).

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy.$$

For flate plate case, $u/U_\infty = f'(\eta)$, and $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$, we obtain

$$\delta_1 = \sqrt{\frac{\nu x}{U_\infty}} \int_{\eta=0}^{\infty} (1 - f'(\eta))\, d\eta = \sqrt{\frac{\nu x}{U_\infty}} \left[\eta_1 - f(\eta_1)\right]$$

where η_1 denotes a point outside boundary layer. From numerical sol^n of Blasius's equation $\eta_1 - f(\eta_1) = 1.7208$, and so.

$$\delta_1 = 1.7208 \sqrt{\frac{\nu x}{U_\infty}}.$$

3. Momentum thickness (δ_2).

The distance by which plate should be moved in normal direction to the wall so that loss of momentum flux caused must be equal to the loss of momentum flux due to the boundary layer.

loss of momentum flux due to plate displacement up to δ_2

$$= \int_0^{\delta_2} (\rho U_\infty w\, dy)\, U_\infty = \rho w U_\infty^2 \delta_2$$

loss of momentum flux due to boundary layer

$$= \int_0^\infty \underbrace{(u \rho w\, dy)\, U_\infty}_{\text{momentum}} - \int_0^\infty \underbrace{(\rho u w\, dy)\, u}_{\text{momentum}}$$

Therefore, by the definition of momentum thickness

$$\rho U_\infty^2 \cancel{y_0} \delta_2 = \cancel{y_0} \cdot \int_0^\infty u(U_\infty - u)\rho dy$$

or, $$\rho U_\infty^2 \delta_2 = \int_0^\infty u(U_\infty - u)\rho dy$$

If fluid is incompressible and body is flat plate at zero incidence then.

$$\boxed{\delta_2 = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy}$$

and $$\delta_2 = \int_0^\infty \sqrt{\frac{\nu x}{U_\infty}}\,(1 - f')\,f' dy$$

or $$\delta_2 = 0.664\sqrt{\frac{\nu x}{U_\infty}}$$ (from numerical solⁿ of Blasius's eqⁿ).

Therefore $$\boxed{\delta_2 < \delta_1 < \delta}$$

Momentum integral equation of boundary layer

OR

von Karman's integral equation

In this topic, we will discuss about the integration of boundary layer equation for two dimensional steady flow of incompressible fluid over a wall. The basic equations are (Prandtl boundary layer equation)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad -(1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad -(2)$$

and $u=0, v=0$ at $y=0$; and $u = U(x)$ when $y \to \infty$. $-(3)$

The outer flow is given by

$$(4) \quad U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx} \quad \text{and} \quad \frac{\partial p}{\partial y} = O(\delta) \text{ in boundary layer so. } \frac{\partial p}{\partial x} = \frac{dp}{dx}$$

Therefore using (4) into (2).

$$u\frac{du}{dx} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

On integration, w.r.to y from $y=0$ to $y=h$, where h is everywhere outside boundary layer, we get

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx}\right)dy = \frac{1}{\rho}\int_0^h \frac{\partial^2 u}{\partial y^2}dy = \frac{1}{\rho}\left[\frac{\partial u}{\partial y}\right]_0^h$$

$$= -\frac{1}{\rho}\left(\frac{\partial u}{\partial y}\right)_0 = -\frac{\tau_0}{\rho}$$

$$\left(\because \frac{\partial u}{\partial y} = 0 \text{ for } y=h\right)$$

Now, using $v = -\int_0^y \frac{\partial u}{\partial x}dy$ from eq^n (1).

In above eq^n, we get

$$\int_0^h \left(u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}dy - U\frac{dU}{dx}\right)dy = -\frac{\tau_0}{\rho}$$

Now,

$$\int_{y=0}^{h} u\frac{\partial u}{\partial x}\,dy - \int_0^h\left(\frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x}\right)dy - \int_0^h U\frac{dU}{dx}\,dy = -\frac{\tau_0}{\rho}$$

or $$\int_0^h u\frac{\partial u}{\partial x}\,dy - [u]_0^h\int_0^y\frac{\partial u}{\partial x}\,dy + \int_0^h u\frac{\partial u}{\partial x}\,dy - \int_0^h U\frac{dU}{dx}\,dy = -\frac{\tau_0}{\rho}$$

or $$\int_0^h\left(2u\frac{\partial u}{\partial x} - U\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right)dy = -\frac{\tau_0}{\rho}$$

or. $$\int_0^h \frac{\partial}{\partial x}\big(u(U-u)\big)\,dy + \frac{dU}{dx}\int_0^h(U-u)\,dy = \frac{\tau_0}{\rho}$$

Now, since, $u = U$ outside boundary layer ie $y \geq h$
so taking $h \to \infty$, we have,

$$\int_0^\infty \frac{\partial}{\partial x}[u(U-u)]\,dy + \frac{dU}{dx}\underbrace{\int_0^\infty(U-u)\,dy}_{\delta_1 U} = \frac{\tau_0}{\rho}$$

using definition of displacement (δ_1) and momentum thickness (δ_2)

we get,

$$\boxed{\delta_1 U\frac{dU}{dx} + \frac{d}{dx}(U^2\delta_2) = \frac{\tau_0}{\rho}}$$

This is momentum integral equation for two-dimensional Incompressible Laminar boundary layers.

Similarly, we can derive the energy integral equation for two dim. laminar boundary layer in Incompressible fluid as

$$\boxed{\frac{d}{dx}[U^3\delta_3] = 2\nu\int_0^\infty\left(\frac{\partial u}{\partial y}\right)^2 dy}$$

where $\delta_3 = \frac{1}{U^3}\int_0^\infty u(U^2-u^2)\,dy$ is energy thickness.