# R-07
# Convex and Nonsmooth Analysis

# Convex Set

A set $C \subseteq \mathbb{R}^n$ is said to be a convex set if for $x, x' \in C$ we have $[x, x'] \subseteq C$, where the closed line segment $[x, x']$ joining $x$ and $x'$ is defined as

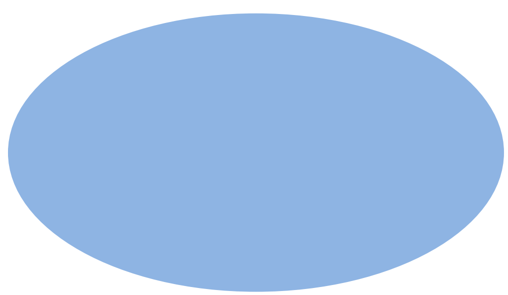
$$[x, x'] := \{tx + (1-t)x' : t \in [0,1]\}.$$

Equivalently, if for $x, x' \in C$ we have $]x, x'[\, \subseteq C$ where

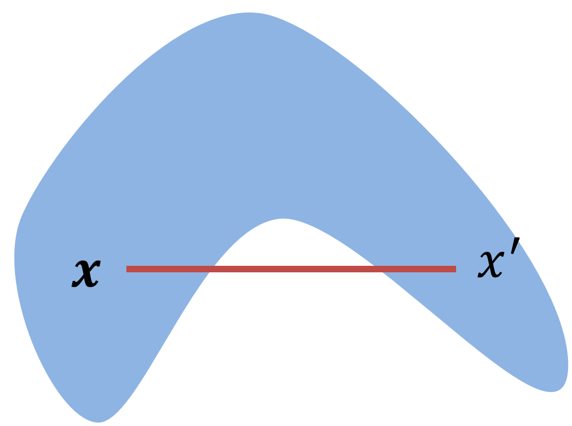
$$]x, x'[\, := \{tx + (1-t)x' : t \in \,]0,1[\}.$$

A set $C$ is convex if

$$x_1, x_2 \in C \implies (1-\lambda)x_1 + \lambda x_2 \in C \text{ for } \lambda \in [0,1].$$



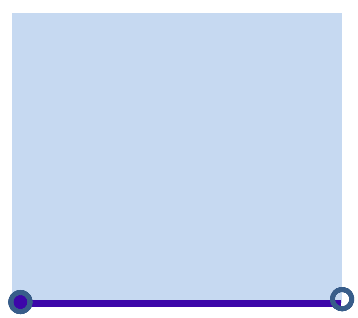
***convex***



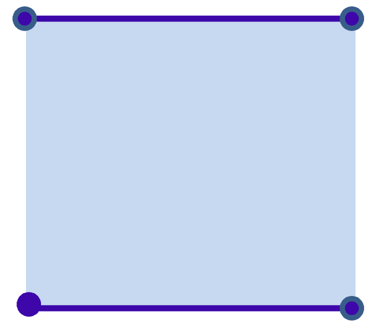

***not convex***

# Convex Set

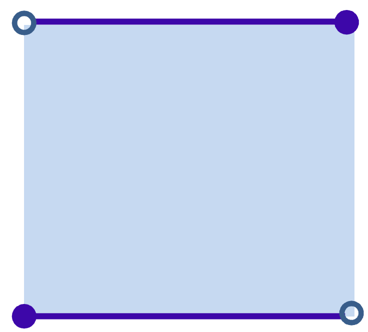
A set $C \subseteq \mathbb{R}^n$ is said to be a *convex set* if for $x, y \in C$ we have $[x, y] \subseteq C$.



*convex*



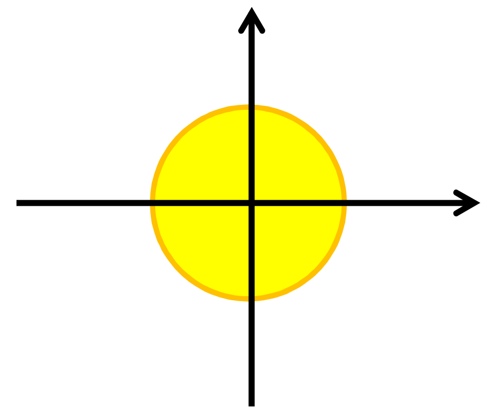
*not convex*



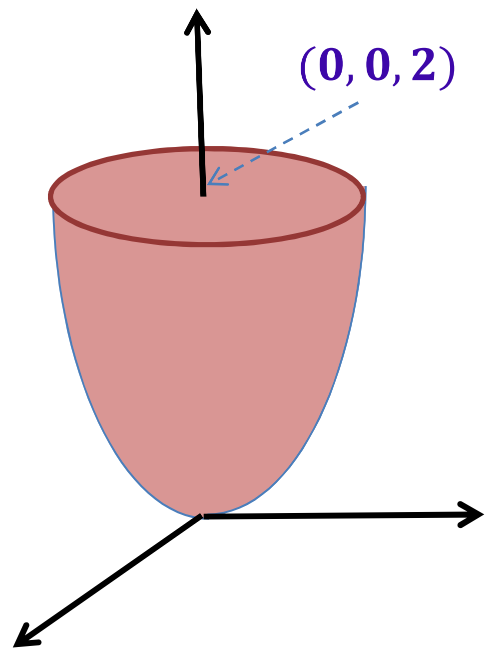
*convex*

# Examples of Convex Sets

$C = \{(x_1, x_2): {x_1}^2 + {x_1}^2 \leq 1\}$.



$C = \{(x, y, z): x^2 + y^2 \leq z, z \leq 2\}$.

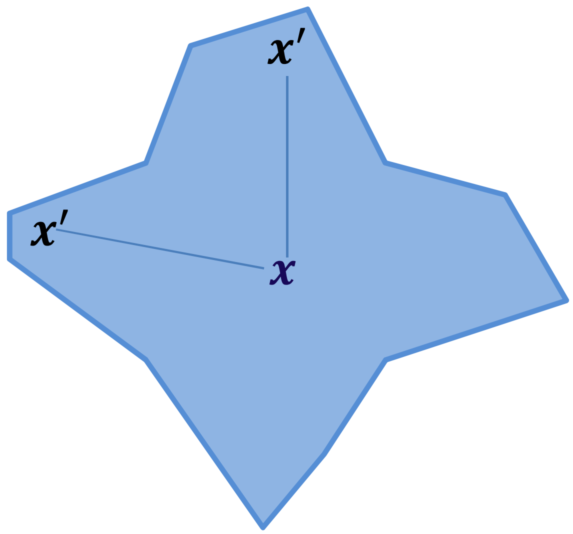



By convention an empty set is a convex set.

# Star-shaped set

A set $C \subseteq \mathbb{R}^n$ is said to be star-shaped set at $x \in C$ if for $x' \in C$ we have

$$(1-\lambda)x + \lambda x' \in C \text{ for } \lambda \in [0,1].$$




A set $C$ is convex if and only if it is star-shaped at every $x \in C$.

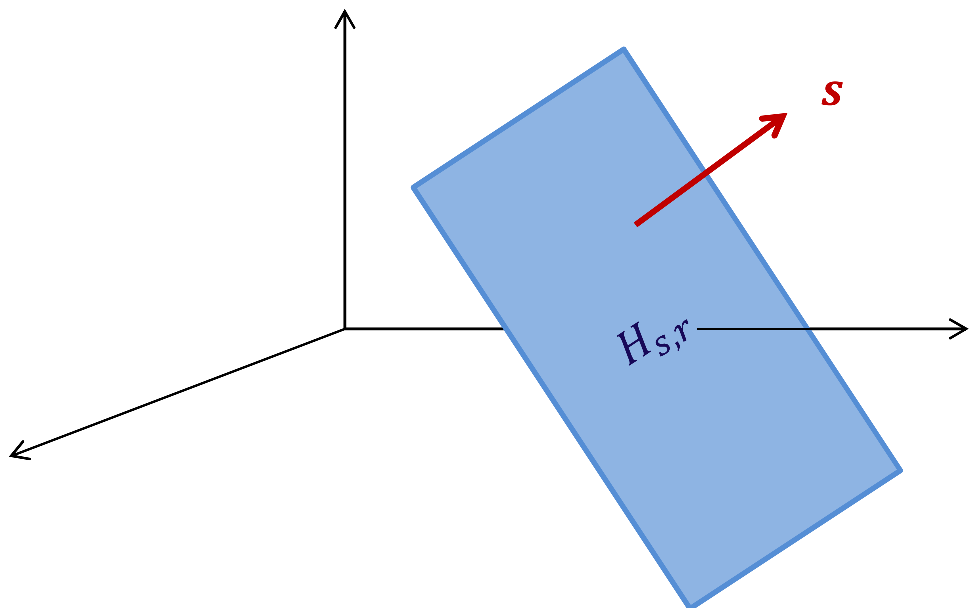
# Hyperplane

A hyperplane associated with $(s, r) \in \mathbb{R}^n \times \mathbb{R}$, $s \neq 0$ is a set denoted by $H_{s,r}$, defined as

$$H_{s,r} := \{x \in \mathbb{R}^n : \langle s, x \rangle = r\}.$$

Hyperplane is a convex set. This hyperplane is parallel to the subspace $H_{s,0}$. The vector $s$ is orthoganal to the hyperplane, that is,

$$H_{s,0} = \{s\}^{\perp}.$$




A hyperplane in $\mathbb{R}^2$ is a line, a plane in $\mathbb{R}^3$ and a point in $\mathbb{R}$.

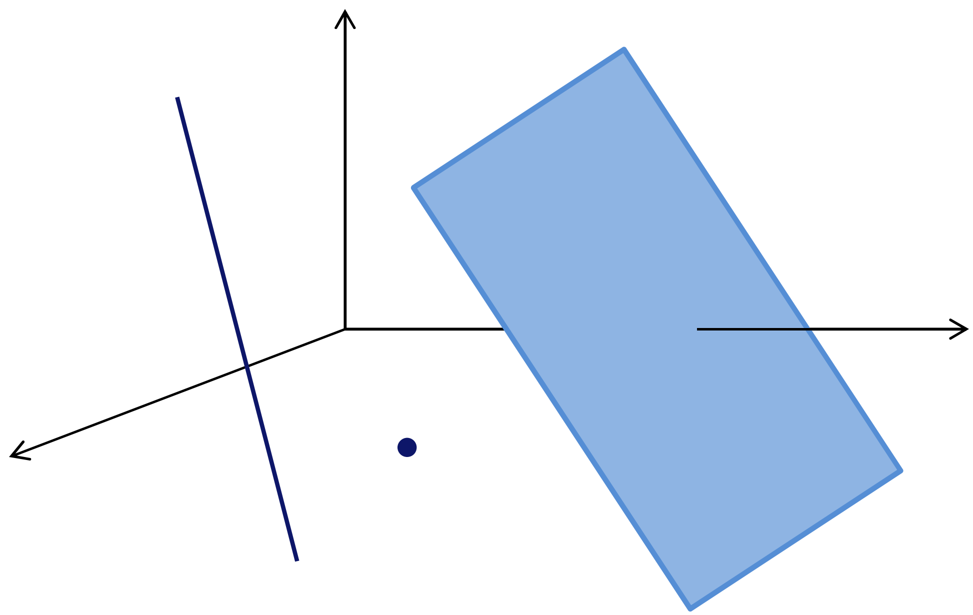
# Affine manifold / Affine subspace / Affine set

A set $V \subseteq \mathbb{R}^n$ is said to be an affine manifold or affine subspace if

$$x_1, x_2 \in V \implies (1-\lambda)x_1 + \lambda x_2 \in V \text{ for } \lambda \in \mathbb{R}.$$

Affine suspaces in $\mathbb{R}^3$ are planes, lines and single points.

A hyperplane is an affine subspace of dimension $n-1$.



# Affine subspaces and Subspaces

*Theorem* 1 If a set $V \subseteq \mathbb{R}^n$ is an affine subspace and if $v \in V$ then $V - v$ is a subspace.

*Proof* Let $S = V - v$. Let $x \in S$ and $\alpha \in \mathbb{R}$. Then there exists $v' \in V$, such that

$$x = v' - v.$$

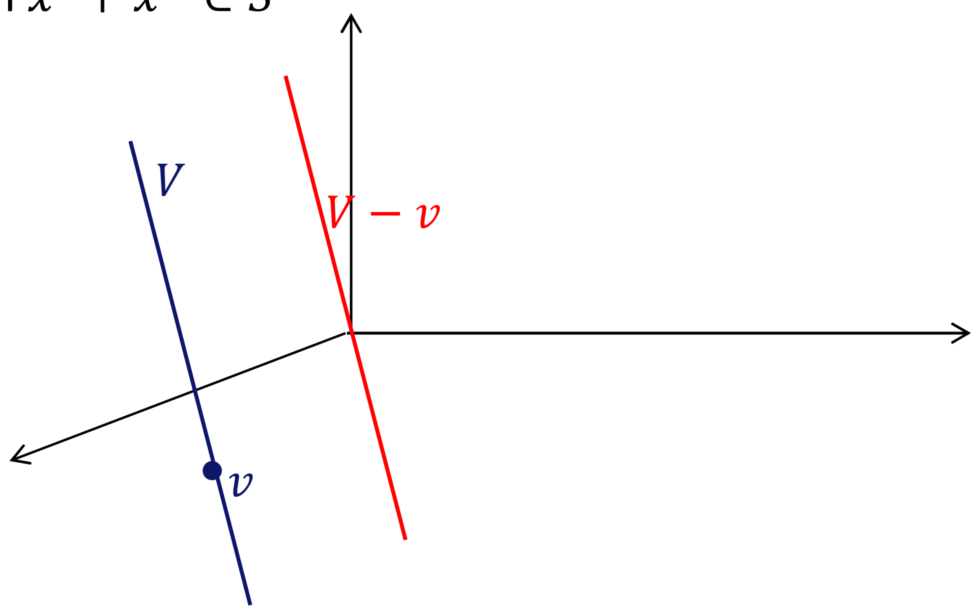
As $V$ is an affine subspace we have $(1-\alpha)v + \alpha v' \in V$. Hence,

$$\alpha x = \alpha(v' - v) = (1-\alpha)v + \alpha v' - v \in S.$$

Let $x', x'' \in S$, then there exist $v', v'' \in V$ such that

$$x' = v' - v, x'' = v'' - v.$$

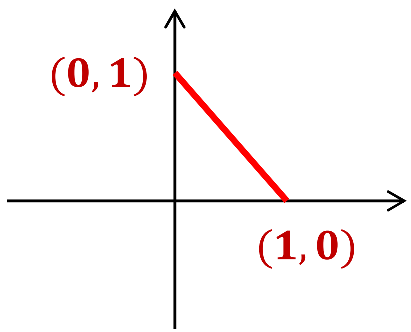
Now $x' + x'' = v' + v'' - 2v = 2\left(\frac{v'}{2} + \frac{v''}{2} - v\right)$. Clearly, $\frac{v'}{2} + \frac{v''}{2} - v \in S$ and hence by the previous justification $x' + x'' \in S$



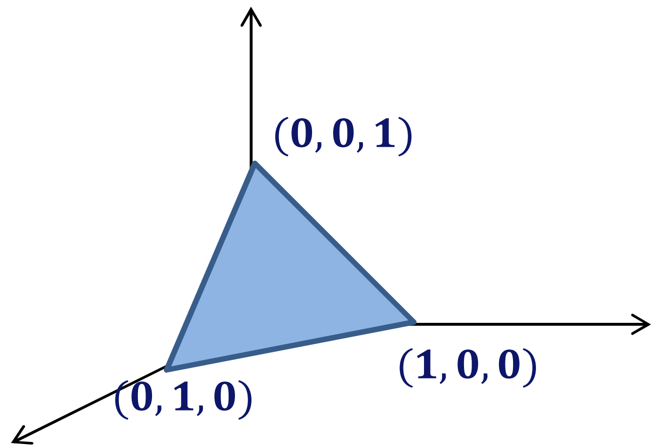
# Simplices

Unit simplex in $\mathbb{R}^k$ is denoted by $\Delta_k$, and is defined as

$$\Delta_k := \left\{\alpha \in \mathbb{R}^k : \sum_{1=1}^{k} \alpha_i = 1,\ \alpha_i \geq 0, i = 1,2, \dots k\right\}.$$




Unit simplex in $\mathbb{R}^2$

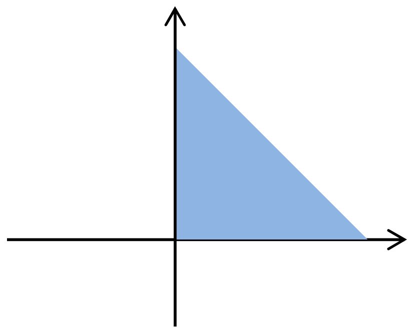



Unit simplex in $\mathbb{R}^3$

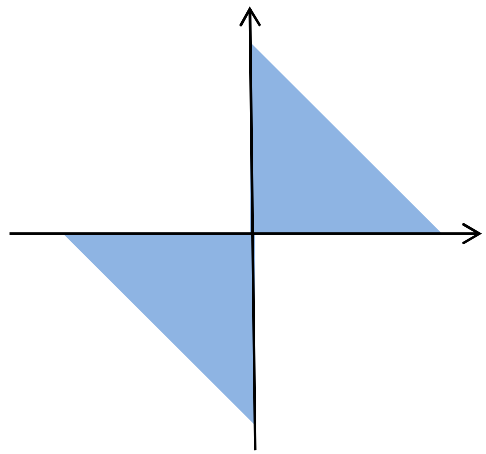
# Cones

A set $K \subseteq \mathbb{R}^n$ is said to be a cone if for $x \in K$ and $\alpha > 0$, we have

$$\alpha x \in K.$$

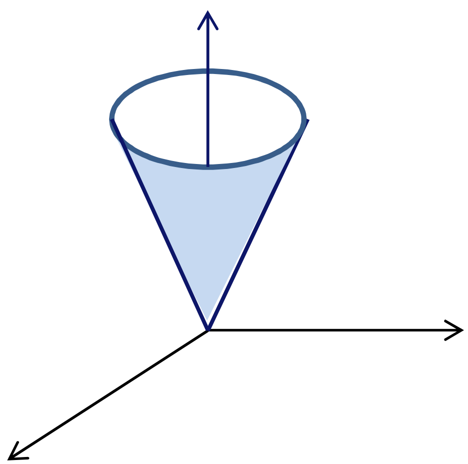


convex cone



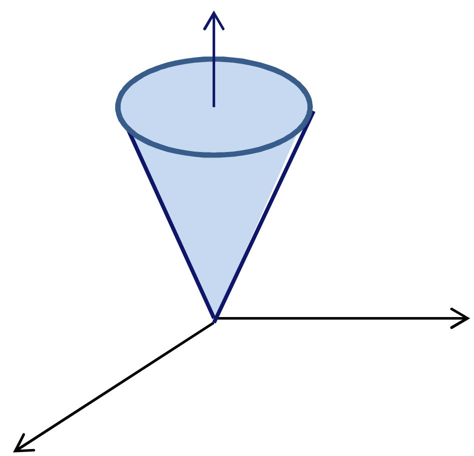
nonconvex cone

# Cones



$$K = \left\{(x_1, x_2, x_3): x_3 = \sqrt{x_1^2 + x_2^2}\right\}$$

nonconvex cone



$$K = \left\{(x_1, x_2, x_3): x_3 \geq \sqrt{x_1^2 + x_2^2}\right\}$$

convex cone

## Properties of Convex Sets

*Lemma* 1 If $\{C_i\}_{i\in I}$ is a family of convex sets in $\mathbb{R}^n$ then $\bigcap_{i\in I} C_i$ is a convex set.

*Proof*. Let $x, y \in \bigcap_{i\in I} C_i$. As $C_i$ is convex we have

$$[x, y] \subseteq C_i \text{ for } i \in I$$

which implies that

$$[x, y] \subseteq \bigcap_{i\in I} C_i.$$

*Lemma* 2 If $C_i, i = 1,2, \ldots, k$, are convex sets then

$$C_1 + C_2 + \cdots + C_k$$

is a convex set.

*Proof*. Let $x, y \in C_1 + C_2 + \cdots + C_k$. Then

$$x = x_1 + x_2 + \cdots + x_k, \quad y = y_1 + y_2 + \cdots + y_k,$$

where $x_i, y_i \in C_i, i = 1,2, \ldots, k$. As each $C_i$ is convex we have

$$(1-\lambda)x_i + \lambda y_i \in C_i, i = 1,2, \ldots, k, \text{ for } \lambda \in [0,1]$$

which implies that for $\lambda \in [0,1]$

$$(1-\lambda)x + \lambda y = \sum_{i=1}^{k}(1-\lambda)x_i + \lambda y_i \in C_1 + C_2 + \cdots + C_k.$$

# Properties of Convex Sets