Q2:- Let f be a mapping of a Banach space X into itself. Let $x_0 \in X$, $r > 0$ and $0 \leq \lambda < 1$. Assume that on the closed ball $B(x_0, r)$ we have

$$\|Fx - Fy\| \leq \lambda \|x - y\| \quad - ①$$

$$\|x_0 - Fx_0\| < (1-\lambda) \cdot r \quad - ②$$

Prove that

① $F^n x_0 \in B(x_0, r)$

② $x^* = \lim F^n x_0$ exist

③ $F(x^*) = x^*$

④ $\|F^n x_0 - x^*\| \leq \lambda^n \cdot r$.

Proof:- Claim: $\langle F^n x_0 \rangle = \{x_0, Fx_0, F^2 x_0, \cdots\} \in B(x_0, r)$.

we will prove it by induction.

Let $n = 0 \Rightarrow F^0 x_0 = I x_0 = x_0$

As $\|x_0 - x_0\| = 0 < r \Rightarrow F^0 x_0 \in B(x_0, r)$.

Let $n = 1$. As $\|Fx_0 - x_0\| < (1-\lambda) \cdot r$ (By given Hypothesis)

As $0 \leq \lambda < 1 \Rightarrow 0 < (1-\lambda) \leq 1$

$\Rightarrow \|Fx_0 - x_0\| < r \Rightarrow Fx_0 \in B(x_0, r)$

$\Rightarrow$ it is true for $n = 1$.

Suppose it is true for $n=k$

ie $\|F^k x_0 - x_0\| < r$ $-(*)$

Let $n=k+1$

Consider $\|F^{k+1} x_0 - x_0\|$

$= \|F^{k+1} x_0 - Fx_0 + Fx_0 - x_0\|$

$\leq \|F^{k+1} x_0 - Fx_0\| + \|Fx_0 - x_0\|$

$= \|F(F^k x_0) - Fx_0\| + \|Fx_0 - x_0\|$

As By $(*)$ $F^k x_0 \in B(x_0, r)$ $\Rightarrow$ By ① and ②

$\leq \lambda \|F^k x_0 - x_0\| + (1-\lambda)\cdot r$

$\leq \lambda \cdot r + (1-\lambda)\cdot r$

$\leq r$

$\Rightarrow F^{k+1} x_0 \in B(x_0, r) \Rightarrow$ it is true for $n=k+1$

$\Rightarrow$ By induction Hypothesis it is true $\forall\, n \in \mathbb{N} \cup \{0\}$.

$\Rightarrow F^n x_0 \in B(x_0, r)$ $\forall\, n \geq 0$.

②:- Claim:- $\lim F^n x_0$ exist.

As $B(x_0, r)$ is a closed subset of Banach space X

$\Rightarrow B(x_0, r)$ is complete. and $F^n x_0 \in B(x_0, r)$. $\Rightarrow$ if we will show that $\langle F^n x_0 \rangle$ is cauchy sequence in $B(x_0, \gamma)$

$\Rightarrow$ it must be cgs in $B(x_0, \gamma)$.

Claim:- $\langle F^n x_0 \rangle$ is cauchy in $B(x_0, \gamma)$.

Let $n, m \in \mathbb{N}$ such that $n > m$.

Consider $\|F^n x_0 - F^m x_0\|$

$= \|F^n x_0 - F^{n-1} x_0 + F^{n-1} x_0 - F^{n-2} x_0 + \cdots + F^{m+1} x_0 - F^m x_0\|$

$\leq \|F^n x_0 - F^{n-1} x_0\| + \|F^{n-1} x_0 - F^{n-2} x_0\| + \cdots + \|F^{m+1} x_0 - F^m x_0\|$

$\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) \|F x_0 - x_0\|$ (using ①)

$= \lambda^m (1 + \lambda + \lambda^2 + \cdots + \lambda^{n-m-1}) \|F x_0 - x_0\|$

$\leq \lambda^m \left(\frac{1 \cdot (1 - \lambda^{n-m})}{(1-\lambda)} \right) (1-\lambda) \cdot \gamma$

$= (\lambda^m - \lambda^n) \cdot \gamma$

As if $m, n \to \infty \Rightarrow \lambda^m, \lambda^n \to 0$

$\Rightarrow \|F^n x_0 - F^m x_0\| \to 0$ as $n, m \to \infty$

$\Rightarrow \langle F^n x_0 \rangle$ is cauchy in $B(x_0, \gamma)$. As $B(x_0, \gamma)$ is complete being a closed subset of a Banach space.

$\Rightarrow \langle F^n x_0 \rangle$ is convergent in $B(x_0, \gamma)$.

ie $\lim F^n x_0$ exist $= x^*$ (say).

(3):- As $F(x^*) = F(\lim F^n x_0)$

$= \lim (F^{n+1} x_0)$ ($\because$ F is continuous being a contraction map on $B(x_0, \gamma)$)

$= x^*$

$\Rightarrow$ $F(x^*) = x^*$

(4):- Claim:- $\|F^n x_0 - x^*\| \leq \lambda^n . \gamma$

we will prove it by induction.

if $n=0$ $\Rightarrow$ $\|F^n x_0 - x^*\| = \|x_0 - x^*\| < \gamma$ ($\because x^* \in B(x_0, \gamma)$)

$\Rightarrow$ it is true for $n=0$.

if $n=1$ $\Rightarrow$ $\|F x_0 - x^*\|$

$= \|F x_0 - F(x^*)\|$ ($\because$ $F(x^*) = x^*$)

As $x_0, x^* \in B(x_0, \gamma)$

$\Rightarrow$ $\|F x_0 - F(x^*)\| \leq \lambda \|x_0 - x^*\|$

$\leq \lambda . \gamma$ ($\because x^* \in B(x_0, \gamma)$)

$\Rightarrow$ it is true for $n=1$

Suppose it is true for $n=k$

ie $\|F^k x_0 - x^*\| \leq \lambda^k . \gamma$ — (*)

Let $n = k+1$

Consider $\|F^{k+1} x_0 - x^*\|$

$= \|F^{k+1} x_0 - F(x^*)\|$

$= \|F(F^k x_0) - F(x^*)\|$

As $F^k x_0 \in B(x_0, \gamma)$, $x^* \in B(x_0, \gamma)$ ⇒ using ① we have

$\leq \lambda \|F^k x_0 - x^*\|$

$\leq \lambda (\lambda^k . \gamma)$ (By *)

$= \lambda^{k+1} . \gamma$

⇒ it is true for $n = k+1$.

⇒ By induction Hypothesis it is true $\forall$ $n \in \mathbb{N} \cup \{0\}$ ~~n ≥ 0~~

⇒ $\|F^n x_0 - x^*\| \leq \lambda^n . \gamma$ $\forall$ $n \in \mathbb{N} \cup \{0\}$.

Q.3:- for what value of λ can we sure that the integral equation $x(t) = \lambda \int_0^1 e^{st} \cos x(s)\, ds + \tan t$ has a continuous solution on $[0,1]$

Solution:- First we recall that the Fredholm integral equation

$$x(t) = \int_0^1 K(s,t,x(s))\, ds + w(t)$$

where $w(t)$ is cts in $[0,1]$ and $K(s,t,r)$ is cts on the domain in $\mathbb{R}^3$ defined by the inequalities

$$0 \le s \le 1, \quad 0 \le t \le 1, \quad -\infty < r < \infty.$$

has unique solution in $C[0,1]$ if K satisfy a Lipschitz condition of the type

$$|K(s,t,\xi) - K(s,t,\eta)| \le \theta|\xi-\eta| \quad (\theta<1)$$

Here $K(s,t,r) = \lambda e^{st}\cos r$ is cts on domain in $\mathbb{R}^3$ defined by inequalities: $0 \le s \le 1$, $0 \le t \le 1$, $-\infty < r < \infty$

and $w(t) = \tan t$ is cts in $[0,1]$

Consider $|K(s,t,\xi) - K(s,t,\eta)|$

$$= |\lambda e^{st}\cos(\xi) - \lambda e^{st}\cos(\eta)|$$

$$\le \lambda e^{st}|\xi-\eta| \le \lambda e|\xi-\eta|$$

$\Rightarrow$ it is required $\lambda e < 1 \Rightarrow \lambda < 1/e$

$\Rightarrow$ for $\lambda \in [0, 1/e)$ we can sure that the integral equation has a cts solution in $[0,1]$