(13) The diameter of metric space X is (1)

$$\operatorname{diam}(X) = \sup\{d(x,y) ; x \in X, y \in X\}$$

this is allowed to be $+\infty$. Show that there cannot exist a surjective contraction a metric space of finite non-zero diameter.

Sol: Let X be a metric space s.t

$$\operatorname{diam}(X) = \sup\{d(x,y) ; x \in X, y \in X\} < \infty$$

say $\operatorname{diam}(X) = k$.

Assume that $\exists$ $f: X \to X$ a contractive surjective map then $\exists \lambda \in [0,1)$ s.t.

$$d(f(x), f(y)) \leq \lambda d(x,y) \quad -(1)$$

Let $x \neq y$ and $x, y \in X$ then $\exists x_1, y_1 \in X$ s.t $f(x_1) = x$ and $f(y_1) = y$. Hence

$$d(x,y) = d(f(x_1), f(y_1)) \leq \lambda d(x_1, y_1) \; [\text{by } (1)] \; -(11)$$

Now since $x_1, y_1 \in X$ similarly $\exists x_2, y_2 \in X$ s.t $f(x_2) = x_1$, $f(y_2) = y_1$

Hence we have

$$d(x_1, y_1) = d(f(x_2), f(y_2))$$
$$\leq \lambda d(x_2, y_2) \; -(111)$$

from (11) and (111) we have

$$d(x,y) \leq \lambda d(x_1, y_1) \leq \lambda^2 d(x_2, y_2)$$

and similarly from induction we have

$$d(x,y) \leq \lambda^n d(x_n, y_n) \text{ where } x_n, y_n \in X$$

$$\Rightarrow d(x,y) \leq \lambda^n k \; -(1V) \; [\because k = \sup\{d(x,y) ; x, y \in X\}]$$

Now as $\lambda \in (0,1)$ then $\lambda^n \to 0$ as $n \to \infty$

Hence from (IV) we get

$d(x,y) \leq \epsilon$ as small as we please

thus $d(x,y) = 0$, so $\underline{x = y}$

which is contradiction as we have choosen distinct points (i.e., $\underline{x \neq y}$).

Hence the claim: that there cannot exist a surjective contraction on a metric space of finite non-zero diameter holds true.

(14) Let X be a Banach space and f a mapping of X into X. Are these two properties of f are equivalent (3)

(i) f has a fixed point

(ii) There is a non-empty set (closed) in X say E such that $f(E) \subset E$ and such that $\|f(x)-f(y)\| < \frac{1}{2}\|x-y\|$ $\forall x,y \in E$.

Pf: Let $f: X \to X$ be a mapping where X is a Banach space and E be closed subset of X.

Let $f(x) = x$ then clearly $f(E) \subseteq E$ [$\because f(x) = x \in E$ $\forall x \in E$]

but

$$\|f(x) - f(y)\| = \|x-y\| > \frac{1}{2}\|x-y\| \ \forall x,y \in E.$$

So there cannot exist a non-empty set E in X s.t. $f(E) \subseteq E$ and $\|f(x)-f(y)\| < \frac{1}{2}\|x-y\|$ $\forall x,y \in E$.

it means (i) $\not\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) As if E is a closed set in X and X is a Banach space, so E is complete.

and it is given that $f(E) \subset E$.

So $f: E \to E$ (can be defined)

Also given that $\|f(x)-f(y)\| < \frac{1}{2}\|x-y\|$ [$\because \theta = \frac{1}{2} \in [0,1)$ for $\|f(x)-f(y)\| \le \theta\|x-y\|$]

So By Banach contraction principle

f has a fixed point (unique)

So (i) holds true

Hence (ii) $\Rightarrow$ (i).

(1f) Let T be a contraction on a metric space

$$d(Tx, Ty) \leq \lambda d(x,y) \quad (\lambda < 1) \quad -(1)$$

Prove that the set $\{x \mid d(x, Tx) \leq \epsilon\}$ is a nonempty, closed, and of diameter atmost $\frac{\epsilon}{1-\lambda}$.

Pf: Let S be the set $\{x \mid d(x,Tx) \leq \epsilon\}$

say $S = \{x : d(x, Tx) \leq \epsilon\}$

Claim: (1) S is non-empty

Let $x_0 \in X$ define iterations on X as (arbitrary)

$x_{n+1} = T(x_n)$ and $Tx_0 = x_1$.

So $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$ [by iterations defined]

$\leq \lambda d(x_n, x_{n-1})$ [by (1)]

$= \lambda d(Tx_{n-1}, Tx_{n-2})$ [by iteration]

$\leq \lambda^2 d(x_{n-1}, x_{n-2})$ [by (1)]

$\vdots$ (proceeding n times)

$\leq \lambda^{n-1} d(x_1, x_0) \to 0$ as $n \to \infty$

[because $\lambda \in [0,1)$]

$\Rightarrow d(x_{n+1}, x_n) \to 0$ as $n \to \infty$

So for $\frac{\epsilon}{2} > 0$ $\exists N \in \mathbb{N}$ such that

$d(x_{n+1}, x_n) < \frac{\epsilon}{2} < \epsilon \quad \forall n \geq N$

$\Rightarrow d(Tx_n, x_n) < \epsilon \quad \forall n \geq N$ [$\because d(Tx_n, x_n) = d(x_{n+1}, x_n)$]

$\Rightarrow x_n \in S$ for $n \geq N$. [$\because$ by definition of members of S]

So S is non-empty. (5)

claim!

(2) S is closed : Let $\{y_n\}$ be sequence in S converging to y.

we will prove that $y \in S$ [$\Rightarrow$ S is closed]

Since $y_n \in S = \{x;\ d(x, Tx) \le \epsilon\}$

$\Rightarrow d(y_n, Ty_n) \le \epsilon$

As, T is contraction map so T is ctr map.

Also metric d is continuous map from $X \times X \to \mathbb{R}$

[Pf! Define $f: X \times X \to \mathbb{R}$ by $f(x, y) = d(x, y)$

Let (x_n, y_n) be sequence in $X \times X$ such that

$(x_n, y_n) \to (x, y)$

then $x_n \to x$ and $y_n \to y$

$\Rightarrow d(x_n, x) \to 0$, $d(y_n, y) \to 0$ as $n \to \infty$

~~$\Rightarrow d(x_n, y_n) \le d(x_n, y) + d(y, y_n)$~~

~~$\le d(x_n, y) +$~~

$\Rightarrow d(x_n, y_n) \le d(x_n, x) + d(x, y_n)$

$\le d(x_n, x) + d(x, y) + d(y_n, y)$

$\to d(x, y)$ as $n \to \infty$

($\because d(x_n, x) \to 0$
$d(y_n, y) \to 0$)

$\Rightarrow d(x_n, y_n) \to d(x, y)$ as $n \to \infty$

$\Rightarrow f(x_n, y_n) \to f(x, y)$

Hence f is ctr map

$\Rightarrow d$ is ctr.]

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Since $d(y_n, Ty_n) \le \epsilon$

taking limit on both sides

$\Rightarrow \lim_{n\to\infty} d(y_n, Ty_n) \le \lim_{n\to\infty} \epsilon$

$\Rightarrow \lim_{n\to\infty} d(y_n, Ty_n) \le \epsilon$

$\Rightarrow d(\lim_{n\to\infty} y_n, \lim_{n\to\infty} Ty_n) \le \epsilon$ [$\because$ d is cts so jointly cts. from $X\times X \to \mathbb{R}$]

$\Rightarrow d(y, \lim_{n\to\infty} Ty_n) \le \epsilon$ [$\because \lim_{n\to\infty} y_n = y$]

$\Rightarrow d(y, T(\lim_{n\to\infty} y_n)) \le \epsilon$ [$\because \lim_{n\to\infty} y_n = y$]

$\Rightarrow d(y, Ty) \le \epsilon$

$\Rightarrow y \in S$

$\Rightarrow$ S is closed

Claim: (3) diameter is atmost $\frac{\epsilon}{1-\lambda}$

$diam(S) = \sup\{d(x,y) ; x,y \in S\}$ for $x,y \in S$

Consider $d(x,y) \le d(x,Tx) + d(Tx,y)$

$\le d(x,Tx) + d(Tx,Ty) + d(Ty,y)$

$\le \epsilon + d(Tx,Ty) + \epsilon$

[$\because x,y \in S$ $\Rightarrow d(x,Tx) \le \epsilon$ $d(y,Ty) \le \epsilon$]

$= 2\epsilon + d(Tx,Ty)$

$\le 2\epsilon + \lambda d(x,y)$ [$\because d(Tx,Ty) \le \lambda d(x,y)$]

$\Rightarrow d(x,y)(1-\lambda) \leq 2t$

$\Rightarrow d(x,y) \leq \dfrac{2t}{1-\lambda}$

$\Rightarrow \sup\limits_{x,y \in S} d(x,y) \leq \dfrac{2t}{1-\lambda}$

$$\boxed{\Rightarrow \operatorname{diam}(S) \leq \frac{2t}{1-\lambda}}$$