Department of Mathematics
MPhil/PhD Coursework Examination
Topics in Analysis

(August 2021)
Time: 3 hrs for attempting the paper + 1 hr for downloading and uploading
Max. Marks : 70
Attempt SEVEN questions in all. All symbols carry their usual meaning.

1. (a) State and prove Banach's Contraction Principle. (6)

   (b) Let $X = l^2$ with its usual norm and the mapping f be given by $f(x) = (1/2, x_1, x_2, x_3, \dots)$ where $x = (x_1, x_2, \dots) \in l^2$. Show that F has no fixed points and state, with justification, why Banach's Contraction Principle does not yield a fixed point in this case? (4)

2. (a) Let $(X, \|\cdot\|)$ be a Banach space and $f : [a, b] \to X$ be a Riemann integrable function. If the function $t \mapsto \|f(t)\|$ is also Riemann integrable, then show that
$$\left\| \int_a^b f(t)\, dt \right\| \leq \int_a^b \|f(t)\|\, dt.$$
(5)

   (b) Is the function $f : (0, 1) \to L^\infty(0, 1)$ Bochner integrable when (a) $f(t) := \chi_{(0,t)}$ (b) $f(t) := tx_0$ where x_0 is a fixed element of $L^\infty(0, 1)$. Justify. (3+2)

3. (a) What is meant by an algebra? Give an example of (a) an algebra of functions which is closed; (b) a collection of functions which is not an algebra. (3)

   (b) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{A}$ be the set of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ where $N \in \mathbb{N}, c_n \in \mathbb{C}, \theta \in \mathbb{R}$. Show that $\mathcal{A}$ is an algebra which separates points of $\mathbb{T}$ and vanishes at no point of $\mathbb{T}$. Further, show that $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\, d\theta = 0$ for $f \in \mathcal{A}$ and deduce that there are continuous functions on $\mathbb{T}$ which are not in the uniform closure of $\mathcal{A}$. Is the Stone Weierstrass theorem applicable here? Give reasons. (7)

4. (a) For a mapping $f : D \to X$ where X is a normed space and D an open subset of X, explain the difference between the statements (a) f' is continuous at x (b) $f'(x)$ is continuous. Prove that if $f'(x)$ exists, then it is continuous and differentiable. (6)

   (b) Let $g \in C[0, 1], \alpha \in \mathbb{C}$ be fixed. Define $F : C[0, 1] \to C[0, 1]$ by $F(x) = \alpha g \cdot x, x \in C[0, 1]$ where $\cdot$ represents pointwise multiplication. Find the derivative of F. (4)

5. (a) Let $f : D \to Y$ be differentiable at $x \in D$ and Z be a normed linear space and $B : Y \to Z$ be a bounded linear operator. Prove that $B \circ f$ is Frechet differentiable at x and find $(B \circ f)'(x)$. (4)

(b) Let $f : H \to \mathbb{R}$ where H is a real Hilbert space and $a \in H$ be fixed. Define $f(x) := \langle a, x\rangle^2, x \in H$. Show that f is differentiable and find its derivative. (6)

6. (a) Let f be defined on an open set Ω in the direct sum space $X = X_1 \oplus X_2 \oplus X_3$, (where each X_j is a Banach space), and take values in a normed space Y, such that all partial derivatives $D_j f$ exist in Ω and are continuous at every point x_0 in Ω. Show that f is Frechet differentiable at x_0 and find an expression for the Frechet derivative of f at x_0. (5)

   (b) Let $\alpha \in [0, 1]$ where $\cos\alpha = \alpha$. Define X to be the space of all continuously differentiable functions on $[0, 1]$ that vanish at α with norm given by $\|x\| = \sup_{0\leq t\leq 1} |x'(t)|$. Prove that there exists a positive number δ such that if $y \in X$, and $\|y\| < \delta$ then there exists an $x \in X$ satisfying $\sin \circ x + x \circ \cos = y$. (5)

7. (a) Let $z_1 = 3 + 4i$, $z_2 = -i$ and $z_3 = \infty$. Compute the spherical distance between (a) z_1, z_2, (b) z_1, z_3 and (c)$\frac{1}{z_1}, \frac{1}{z_3}$. Show further that spherical distance defines a metric d_∞ on the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. (3 + 2)

   (b) Let d denote the usual distance metric on $\mathbb{C}$ and d_∞ the spherical metric on $\mathbb{C}_\infty$. For $a \in \mathbb{C}$ designate $B(a, \epsilon)$ an open ball in $(\mathbb{C}, d)$ and by $B_\infty(a, \epsilon)$ an open ball in $(\mathbb{C}_\infty, d_\infty)$. Show that if $a \in \mathbb{C}$ and $r > 0$ then there exists a $\rho > 0$ such that $B_\infty(a, \rho) \subset B(a, r)$. Conversely, show that if $a \in \mathbb{C}$ and $\rho > 0$ then there exists $r > 0$ such that $B(a, r) \subset B_\infty(a, \rho)$. (5)

8. (a) Show that if f is a meromorphic function in a domain D then it is continuous in D with respect to the spherical distance. (4)

   (b) Let $\{f_n\}$ be a sequence of meromorphic functions defined in a domain D. Show that $\{f_n\}$ converges uniformly with respect to spherical distance on compact subsets of D to a function f if and only if about each z_0 there is a closed disc $B(z_0, r)$ about z_0 of radius $r > 0$ in which $|f_n - f| \to 0$ or $|\frac{1}{f_n} - \frac{1}{f}| \to 0$ uniformly as $n \to \infty$. (6)

9. (a) When is a family of meromorphic functions called normal? Give non-trivial examples of (i) a normal family of holomorphic function (ii) a normal family of meromorphic functions. (5)

   (b) Consider the statement "A family of normal holomorphic functions which is locally uniformally bounded in a domain D is normal in D." State an analog of this result for a family of meromorphic functions. Check whether or not the family $\{f_n\}$ where $f_n(z) = \frac{z}{(z-1/n)}$ is normal on the disc of radius 1 centered at 2? (5)

10. (a) Let $\mathcal{F}$ be a normal family of holomorphic functions in a domain D satisfying $\min_{z\in\sigma} |f(z)| \leq M \; \forall f \in \mathcal{F}$ where σ is a bounded and closed subset of D and M is a positive number. Show that the family $\mathcal{F}$ is uniformly bounded on each bounded closed subset E of D. Illustrate this result with a non-trivial example. (6)

    (b) Let $f_n(z) := 1 + 3^n z + 3^{2n} z^2$ and $g_n(z) = \frac{z}{(z - \frac{1}{3^n})}$. What can you deduce about the normality of the families $\{f_n\}$ and $\{g_n\}$ in the domain $D = \{z : |z| < 3\}$. Justify. (4)