Lecture 9.

Theorem 1.23 Second Version of the Implicit Function Th.

Let X, Y, Z be normed sp. Y, complete, $\Omega \subseteq X \times Y$ is open. Let $(x_0, y_0) \in \Omega$. Assume

(i) F is cts at (x_0, y_0).

(ii) $F(x_0, y_0) = 0$

(iii) $D_2 F$ exists in Ω.

(iv) $D_2 F$ is cts at (x_0, y_0)

(v) $D_2 F(x_0, y_0)$ is invertible.

- F is continuously diff in Ω.

LO2

Then $\exists$ a fnc f defined on a nbd W of x_0, st.

- $(x, f(x)) \in \Omega$ if $x \in W$
- $F(x, f(x)) = 0$, $x \in W$
- $f(x_0) = y_0$.
- f is cts at x_0
- f is unique.
- f is continuously differentiable

and.

$$f'(x) = -D_2F(x, f(x))^{-1} \circ D_1F(x, f(x))$$

$x \in W$.

Pf: Left as an assignment.

What does F continuously diff mean?

$F : \Omega \longrightarrow Z$.

This means, that

(a) $F'(a)$ exists for each $x \in \Omega$.

(b) the map $F' : \Omega \longrightarrow \mathcal{L}(X \times Y, Z)$ is continuous.

L04

Theorem 1.24

Write Theorem 1.23, when $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $Z = \mathbb{R}^m$.

Theorem 1.25 Inverse Function Theorem I:

Let f be a continuously differentiable map from Ω, an open subset of X ~~is~~ a Banach sp., to Y, a normed linear sp

$\left(f : \Omega \subseteq X \longrightarrow Y. \right)$ L05

Let $x_0 \in \Omega$.

If $f'(x_0)$ is invertible, then there is a continuously diffble function g, defined on a nbd N of $f(x_0)$, such that

$$\underline{f(g(y)) = y} \quad \forall\, y \in N$$

<u>Pf</u>: Define $F : \Omega \times Y \longrightarrow Y$ by $F(x, y) = f(x) - y$.

L06.

for $x_0 \in \Omega$.

let $y_0 = f(x_0)$

Then

- $F(x_0, y_0) = 0$
- F is diff on $\Omega \times Y$??

let $(x,y) \in \Omega \times Y$.

Take $h = (h_1, h_2) \in \Omega \times Y$.

Then

$$F((x,y)+h) - F(x,y)$$

$$= F(x+h_1, y+h_2) - F(x,y)$$

$$= f(x+h_1) - y - h_2 - f(x) + y$$

$$= f(x+h_1) - f(x) - h_2 .$$

Choose A to be

$$Ah = A(h_1, h_2) = f'(x)h_1 - h_2 .$$

Then

$$\frac{\| F((x,y)+h) - F(x,y) - Ah \|_{Y}}{\|h\|} .$$

$$= \frac{\| f(x+h_1) - f(x) - f'(x)h_1 \|}{\|(h_1, h_2)\|_{X \times Y}}$$

$$\leq \frac{\| f(x+h_1) - \cdots \|}{\|h_1\|} \longrightarrow 0$$

as f is diff on Ω.

$\therefore$ $F'(x,y) = f'(x) - I_Y$

$\forall\, (x,y) \in \Omega \times Y$.

Moreover

$(x,y) \longmapsto f'(x) - I_Y$

is cts from $\Omega \times Y$ to Y.

- Thus F is continuously differentiable on $\Omega \times Y$.

109.

$$(D_1 F)(x,y) = f'(x)$$
$\forall\, x \in \Omega$.
$y \in Y$.

- $\therefore D_1 F$ exists on $\Omega \times Y$
- $(D_1 F)(x_0, y_0) = f'(x_0)$

which is given to be invertible.

- Since f is continuously diff, it follows that $D_1 F$ is cts on (x_0, y_0).

$\therefore$ By the 2nd version of the I.F.T., $\exists$ a nbd $\mathcal{V}$ of $y_0 = f(x_0)$

and a continuously diff function g, such that

- $(g(y), y) \in \Omega \quad \forall y \in \mathcal{V}$
- $F(g(y), y) = 0 \quad \forall y \in \mathcal{V}$.
- $g(y_0) = x_0$.

$\to \; f(g(y)) - y = 0 \quad \forall y \in \mathcal{V}$.

$f(g(y)) = y$ $\square$

on $\mathcal{N}$

<u>Th 1.26</u>: Surjection Mapping Th. 1.

Let X, Y be Banach sp.

$\Omega \subseteq X$, open. Let

$f : \Omega \longrightarrow Y$ be a

continuously diff map.

Let $x_0 \in \Omega$ and $y_0 = f(x_0)$.

If $f'(x_0)$ is invertible then $f(\Omega)$ is a nbd of y_0.

<u>Pf</u>: Follow proof of previous Th. to obtain g, N.

$f(g(y)) = y \quad \forall y \in N$.

Thus every $y \in N$ is the image of some point in Ω, namely y.

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