# Department of Mathematics

# MPhil/PhD Coursework Supplementary Examination

# **Topics in Analysis**

(November 2022)

Time: 3 hr

Max. Marks : 70

Attempt SEVEN questions in all. All symbols carry their usual meaning.

1. Let $K : [a,b] \times [a,b] \to \mathbb{R}$ be continuous. Find conditions on $\lambda \in \mathbb{R}$ such that the integral equation $x(t) = \lambda \int_a^t K(t,\tau)x(\tau)\,d\tau$ has a unique solution in $x \in C[a,b]$. (10)

2. State and prove Banach's Contraction Principle. Let $X = c_0$, the Banach Space of all real sequences that tend to zero, equipped with norm $\|x\| := \max_i |x_i|$ and suppose that the mapping $f$ is given by $f(x) = (1,1,1,x_1,x_2,x_3,\ldots)$ where $x = (x_1,x_2,\ldots) \in X$. Verify if the conclusion of the above theorems holds for $f$ and if not, give reasons, justifying your claims in detail. (10)

3. (a) Let $X, Y$ be Banach spaces and $f : [a,b] \to X$ . When is $f$ said to be Riemann integrable ? Show that if $f$ is continuous it is Riemann integrable. (6)

   (b) Let $(M, \Sigma, \mu)$ be measure space and $X$ a Banach space. When is a function $f : M \to X$ said to be strongly measurable ? When is a strongly measurable function $f : M \to X$ Bochner measurable and how is its Bochner integral defined? (4)

4. State Stone-Weierstrass's Theorem for an algebra of complex continuous functions defined on a compact set $K$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{A}$ be the set of all functions of the form $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ where $N \in \mathbb{N}, c_n \in \mathbb{C}, \theta \in \mathbb{R}$. Determine which of the conditions of the above mentioned result are satisfied by $\mathcal{A}$. Further check whether or not the conclusion of the theorem holds? Give a reason in either case. (2+8)

5. (a) Let $F : C[0,1] \to C[0,1]$ given by $(F(\phi))(t) = \int_0^t \phi^2(s)\,ds$. Find the Frechet derivative of $F$ at any $\phi_0 \in C[0,1]$. (6)

   (b) Let $L$ be a bounded linear operator on a real Hilbert space $H$ and define $F : H \to \mathbb{R}$ by $F(x) = \langle x, Lx\rangle + \langle x, x\rangle$. Find the Frechet derivative of $F$ at $x_0 \in H$. (4)

6. (a) Let $f$ be defined on an open set $\Omega$ in the direct sum space $X = X_1 \oplus X_2 \oplus X_3 \ldots X_n$, (where each $X_j$ is a Banach space), and take values in a normed space $Y$, such that all partial derivatives $D_j f$ exist in $\Omega$ and are continuous at every point $x_0$ in $\Omega$. Show that $f$ is Frechet differentiable at $x_0$ and find an expression for the Frechet derivative of $f$ at $x_0$. (5)

   (b) Let $f : \Omega \to Y$ be Frechet differentiable, where $\Omega \subset X$, is open, $X, Y$ are open and suppose $a, b, x_0 \in \Omega$ with the line segment $S$ joining $a$ to $b$ contained in $\Omega$. Show that $\|f(b) - f(a) - f'(x_0)(b-a)\| \leq \|b-a\| \sup_{x\in S} \|f'(x) - f'(x_0)\|$. (5)

7. State and prove an Inverse Function Theorem for a function $f : \Omega \to Y$ where $\Omega \subset X$ is an open set and $X, Y$ are Banach spaces. You may clearly state and use an appropriate implicit function theorem if required. (10)

8. Define convolution $f * g$ for $f \in L^p(\mathbb{R}^N), g \in L^1(\mathbb{R}^N), 1 \leq p < \infty$. Show further that $f * g \in L^p(\mathbb{R}^N)$ and $\|f * g\|_p \leq \|f\|_p \|g\|_1$. (10)

9. (a) Let $\Omega \subset \mathbb{R}^N$ be open. When is a function $f \in L^1_{loc}(\Omega)$ said to be weakly differentiable? Show that for $\Omega = (-1, 1)$ the function $f(t) = |t| + t$ is weakly differentiable and find its weak derivative. (4)

   (b) Show that if $f \in L^1_{loc}(\Omega)$ is weakly differentiable with weak derivative of $f$ zero, then $f =$ constant. (6)

10. Let $\Omega = (a, b) \subset \mathbb{R}$. Define the spaces $W^{1,p}(a, b), 1 \leq p < \infty$. Define an inner product on $W^{1,2}(a, b)$ that makes it a Hilbert space, giving full justification. (10)