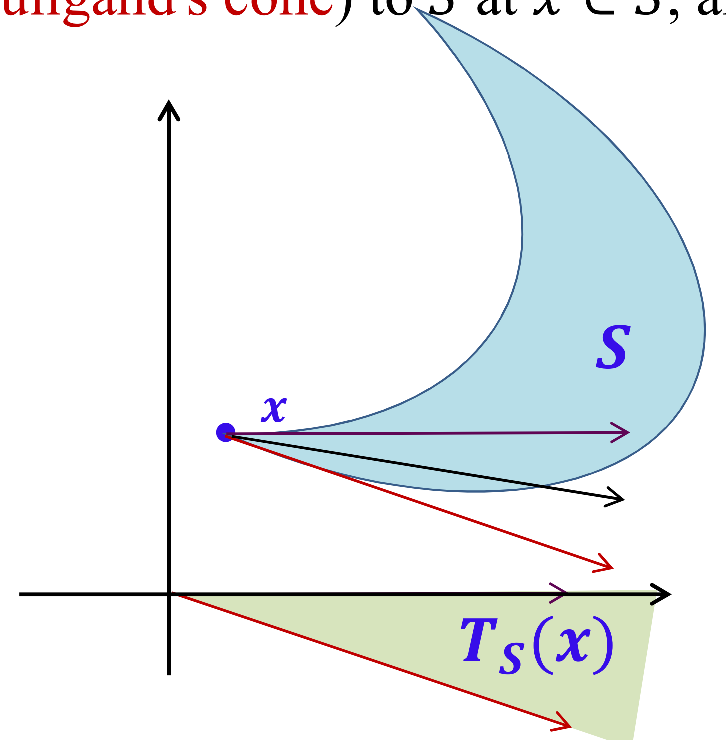
R-07
Convex and Nonsmooth Analysis

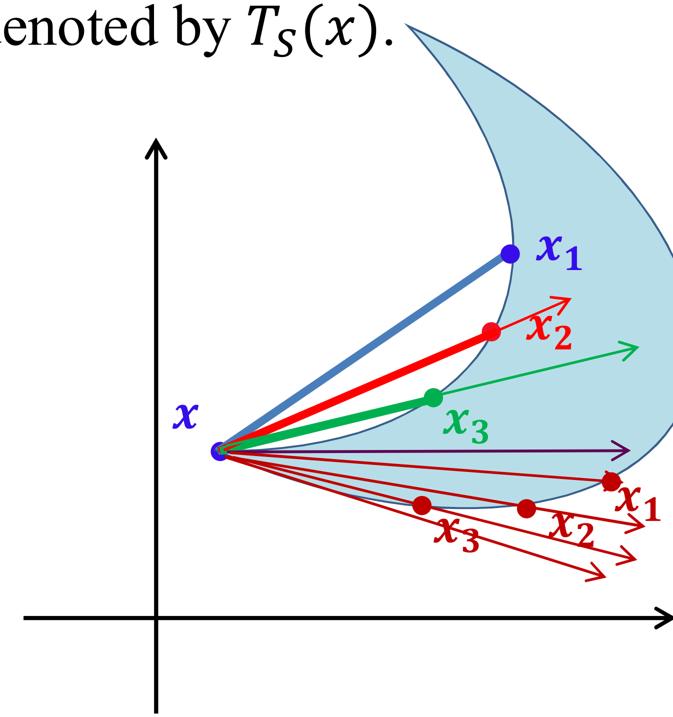
Tangent Direction and Tangent Cone

Let $S \subseteq \mathbb{R}^n$ be a nonempty set. A direction $d \in \mathbb{R}^n$ is said to be a tangent direction to S at $x \in S$ if there exist a sequence $\{x_k\} \subseteq S$ and a sequence $\{t_k\}$ such that, when $k \to \infty$,

$$x_k \to x,\ t_k \downarrow 0, \frac{x_k - x}{t_k} \to d.$$

The set of all such directions is called the tangent cone (contingent cone, or Bouligand's cone) to S at $x \in S$, and is denoted by $T_S(x)$.





Tangent Cone

i) Clearly, $0 \in T_S(x)$.

ii) If $d \in T_S(x)$ then $\alpha d \in T_S(x)$ for $\alpha > 0$.

If $d \in T_S(x)$ then there exists a sequence $\{x_k\} \subseteq S$ and a sequence $\{t_k\}$ such that, when $k \to \infty$,

$$x_k \to x,\ t_k \downarrow 0, \frac{x_k - x}{t_k} \to d.$$

Let $r_k = \frac{t_k}{\alpha}$. As $t_k \downarrow 0$ we have $r_k \downarrow 0$ and

$$\frac{x_k - x}{r_k} = \alpha\left(\frac{x_k - x}{t_k}\right) \to \alpha d.$$

iii) If $x \in \text{int}S$ then $T_S(x) = \mathbb{R}^n$.

Clearly, $T_S(x) \subseteq \mathbb{R}^n$.

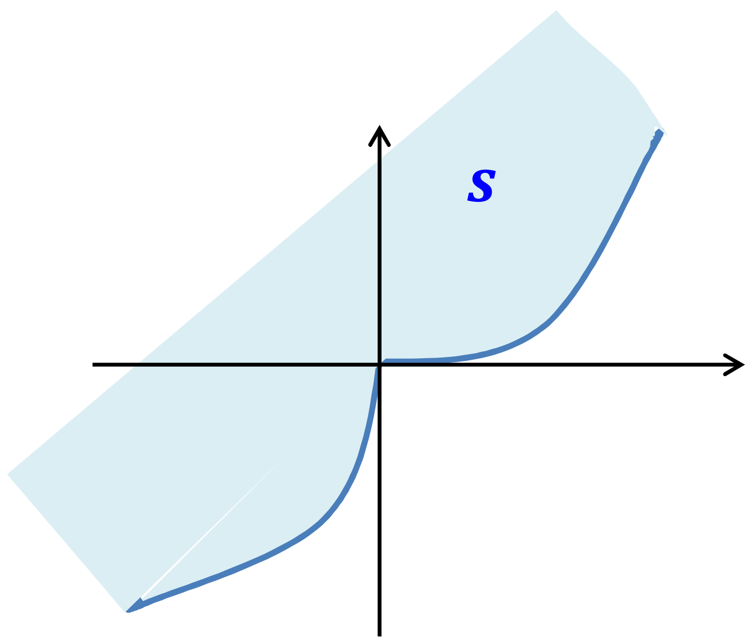
Let $d \in \mathbb{R}^n$. Since $x \in \text{int}S$ there exists $\delta > 0$ such that $B_\delta(x) \subseteq S$. Clearly,

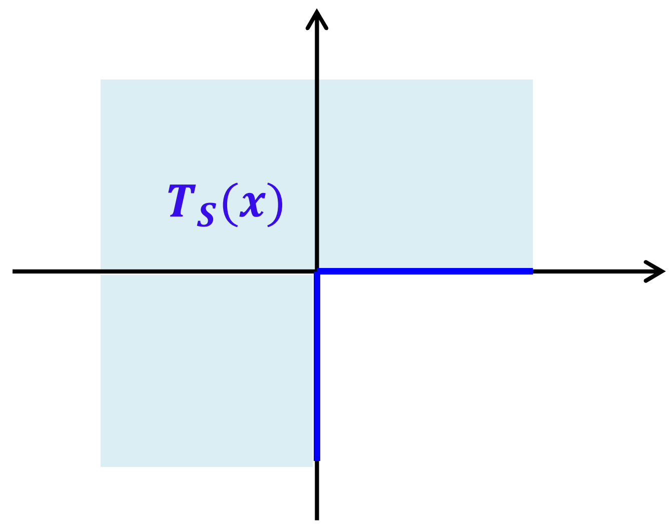
$$x_k = x + \frac{\delta d}{2k\|d\|} \in B_\delta(x) \subseteq S.$$

Let $t_k = \frac{\delta}{2k\|d\|}$. Then $t_k \downarrow 0$ and $\frac{x_k - x}{t_k} = d \to d$.

Thus, $d \in T_S(x)$, and hence $\mathbb{R}^n \subseteq T_S(x)$.

Tangent cone is not necessarily convex





Equivalent Definition

Proposition A direction d is tangent to S at $x \in S$ if and only if there exist a sequence $\{d_k\} \subseteq \mathbb{R}^n$ and a sequence $\{t_k\}$ such that, when $k \to \infty$,

$$d_k \to d,\ t_k \downarrow 0,\ x + t_k d_k \in S, \text{ for all } k.$$

Proof Let $d \in T_S(x)$, then there exist a sequence $\{x_k\} \subseteq S$ and a sequence $\{t_k\}$ such that, $x_k \to x$, $t_k \downarrow 0$, $\frac{x_k - x}{t_k} \to d$. Define

$$d_k = \frac{x_k - x}{t_k}, \text{ for all } k.$$

Then $d_k \to d$ and $x + t_k d_k = x_k \in S$, for all k.

Conversely, let there exist a sequence $\{d_k\} \subseteq \mathbb{R}^n$ and a sequence $\{t_k\}$ such that, $d_k \to d$, $t_k \downarrow 0$, $x + t_k d_k \in S$, for all k. Define

$$x_k = x + t_k d_k, \text{ for all } k.$$

Then $\{x_k\} \subseteq S$ and $\frac{x_k - x}{t_k} = d_k \to d$.

Tangent cone is closed

Proposition The tangent cone is a closed set.

Proof Let $\{d_l\} \subseteq T_S(x)$ be such that $d_l \to d$. For each d_l there exist a sequence $\{x_l^k\} \subseteq S$ and a sequence $\{t_l^k\}$ such that, for $k \to \infty$,

$$x_l^k \to x,\ t_l^k \downarrow 0, \frac{x_l^k - x}{t_l^k} \to d_l.$$

For each $l > 0$ we can find $\bar{k}_l \in \mathbb{N}$ such that

$$\left\| \frac{x_l^k - x}{t_l^k} - d_l \right\| < \frac{1}{l} \qquad \text{for all } k \geq \bar{k}_l.$$

For $l = 1$, in particular for $k = k_1 = \bar{k}_1$ we have

$$\left\| \frac{x_1^{k_1} - x}{t_1^{k_1}} - d_1 \right\| < 1.$$

For $l = 2$, in particular for $k = k_2 = \max\{k_1, \bar{k}_2\}$ we have

$$\left\| \frac{x_2^{k_2} - x}{t_2^{k_2}} - d_2 \right\| < \frac{1}{2}.$$

For $l = 3$, in particular for $k = k_3 = \max\{k_2, \bar{k}_3\}$ we have

$$\left\| \frac{x_3^{k_3} - x}{t_3^{k_3}} - d_3 \right\| < \frac{1}{3}.$$

continued

Proceeding like this we get

$$\left\| \frac{x_l^{k_l} - x}{t_l^{k_l}} - d_l \right\| < \frac{1}{l}$$

where $k_{l+1} \geq k_l$ for all l. As $x_l^k \rightarrow x$ and $k_{l+1} \geq k_l$ for all l it follows that $x_l^{k_l} \rightarrow x$ as $l \rightarrow \infty$. Similarly, $t_l^{k_l} \downarrow 0$ as $l \rightarrow \infty$.

Given $\varepsilon > 0$ there exists $\hat{l} \in \mathbb{N}$ such that $\frac{1}{\hat{l}} < \frac{\varepsilon}{2}$. Hence

$$\left\| \frac{x_l^{k_l} - x}{t_l^{k_l}} - d_l \right\| < \frac{\varepsilon}{2}, \qquad \forall\, l \geq \hat{l}.$$

As $d_l \rightarrow d$ there exists $\bar{l} \in \mathbb{N}$ such that

$$|d_l - d| < \frac{\varepsilon}{2}, \qquad \forall\, l \geq \bar{l}.$$

Let $\tilde{l} = \max\{\hat{l}, \bar{l}\}$. Then

$$\left\| \frac{x_l^{k_l} - x}{t_l^{k_l}} - d \right\| < \varepsilon, \qquad \forall\, l \geq \tilde{l}.$$

Hence there exist a sequence $\{x_l^{k_l}\} \subseteq S$ and a sequence $\{\, t_l^{k_l}\}$ such that, for $l \rightarrow \infty$,

$$x_l^{k_l} \rightarrow x,\ t_l^{k_l} \downarrow 0, \frac{x_l^{k_l} - x}{t_l^{k_l}} \rightarrow d$$

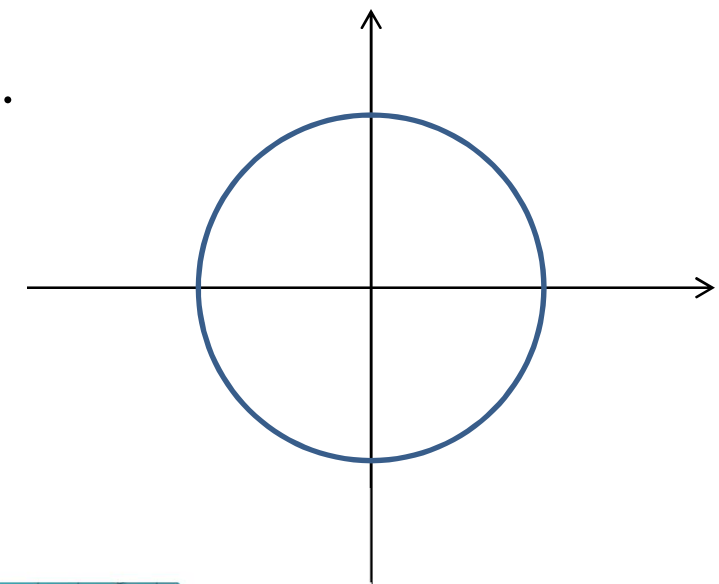
which implies that $d \in T_S\ (x)$.

Distance function

Let $S \subseteq \mathbb{R}^n$ be a nonempty set. A function $d_S : \mathbb{R}^n \to \mathbb{R}$ defined as

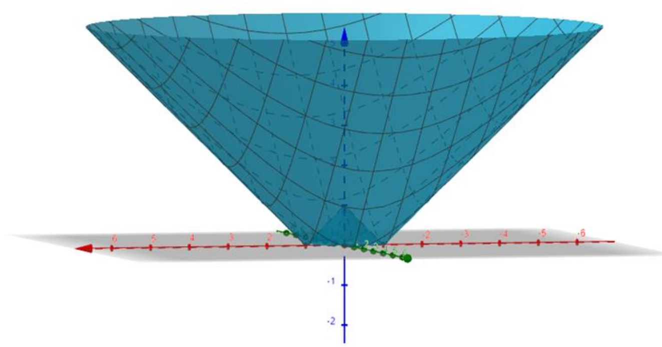
$$d_S(x) = \inf_{x \in S} \|y - x\|$$

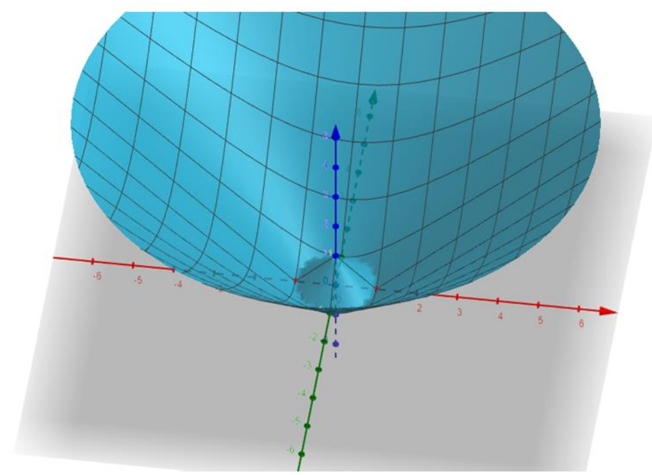
is called distance function.



Example Let $S = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$.

$$d_S(x_1, x_2) = \left| 1 - \sqrt{x_1^2 + x_2^2} \right|$$





Example

Example Let $S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$.

$$d_S(x_1, x_2) = \begin{cases} 0, & \text{if } x_1^2 + x_2^2 \leq 1, \\ \sqrt{x_1^2 + x_1^2} - 1, & \text{if } x_1^2 + x_2^2 > 1. \end{cases}$$