Ex. 3.1.2

given pde $u_t + u u_x = 0$

with $u_0(x) = \begin{cases} 0 & ; \ x \le 0 \\ x/\alpha & ; \ 0 < x \le \alpha \\ 1 & ; \ x > \alpha \end{cases}$

for char. $\frac{dx}{dt} = u(x,t)$ and $u_t + u u_x = \frac{du}{dt} = 0$

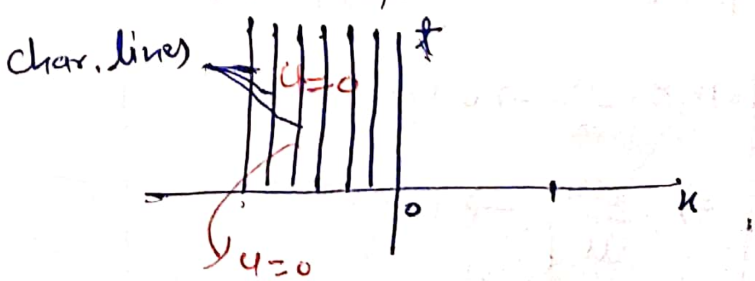
Now check characteristic equation passing through a point

(i) $\left.\frac{dx}{dt}\right|_{(x_0,0)} = u(x_0,0) = 0$ if $x_0 \le 0$

$\Rightarrow \frac{x - x_0}{t - 0} = 0 \Rightarrow x = x_0$ ( equation of char. eqn passing through $(x_0, 0)$ }

and along every char. line $u = 0$ initially, and remain constant over the line

So, we can conclude that $u(x,t) = 0$ for $x \le 0$



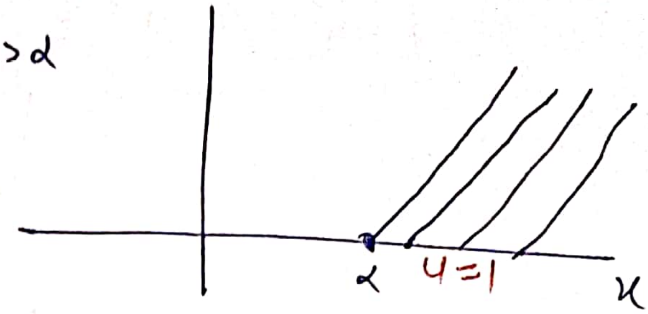

Now check for $x > \alpha$

$\frac{dx}{dt} = u(x,t) \Rightarrow \left.\frac{dx}{dt}\right|_{(x_0,0)} = u(x_0,0) = 1$ for $x_0 > \alpha$

$\Rightarrow \quad \frac{x-x_0}{t-0} = u(x_0,0) = 1 \quad \text{for } x_0>\alpha$

$\Rightarrow \quad x = t + x_0, \quad x_0 > \alpha$

represents lines with slope 1




Since initially $u=1$ on the char. lines for $x>\alpha$

$\Rightarrow$ $U(x,t)$ remains same over the lines

$\Rightarrow \quad u(x,t) = 1 \quad \text{for} \quad x > \alpha + t$

(III) $\quad \left.\frac{dx}{dt}\right|_{(x_0,0)} = u(x_0,0) = \frac{x_0}{\alpha} \quad \text{for } 0<x<\alpha.$

$\Rightarrow \quad x - x_0 = \frac{x_0}{\alpha} t. \Rightarrow \boxed{x = \frac{x_0 t}{\alpha} + x_0}$ → represents Char. equation with slope $\frac{x_0}{\alpha}$

$\Rightarrow \quad x_0\left(\frac{t}{\alpha}+1\right) = x \quad \Rightarrow \boxed{x_0 = \frac{x\alpha}{t+\alpha}}$ — (***)

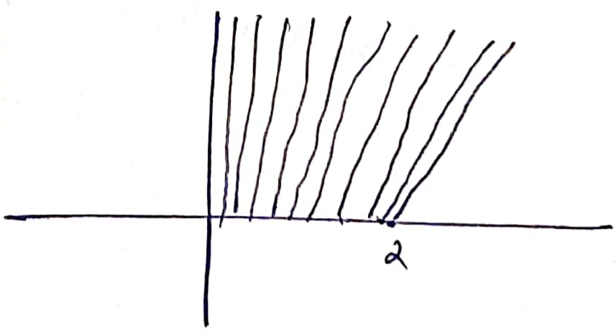
Note:- Since $\frac{x_0}{\alpha} \to 1$ as $x_0 \to \alpha$

and $\frac{x_0}{\alpha} \to 0$ as $x_0 \to 0$

$\Rightarrow \quad \left.\frac{dx}{dt}\right|_{(x_0,0)} \to 1$ as $x_0 \to \alpha$

$\left.\frac{dx}{dt}\right|_{(x_0,0)} \to 0$ as $x_0 \to 0$

We can observe the slope of characteristics with t-Axis




Here $u(x,t) = \frac{x_0}{\alpha}$ if $0 < x \le \alpha + t$

$= \frac{x}{t+\alpha}$

Since we have to maintain the slope $\frac{x_0}{\alpha}$ in between 0 and 1 so,

$$0 < \frac{x_0}{\alpha} < 1$$

$$\Rightarrow \quad 0 < \frac{x}{t+\alpha} < 1$$

$$\Rightarrow \quad 0 < x < t+\alpha$$

So, $u(x,t) = \frac{x_0}{\alpha} = \frac{x}{t+\alpha}$ if $0 < x < \alpha + t$

Now. Combining all steps, we get

$$u(x,t) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x}{t+\alpha} & \text{if } 0 < x < \alpha + t \\ 1 & \text{if } x > \alpha + t \end{cases}$$

Ep 2.1.3

given pde $u_t + u u_x = 0$

with $u_0(x) = \begin{cases} 1 & x \le 0 \\ 1 - x/t & 0 < x \le t \\ 0 & x > t \end{cases}$

To calculate char. lines

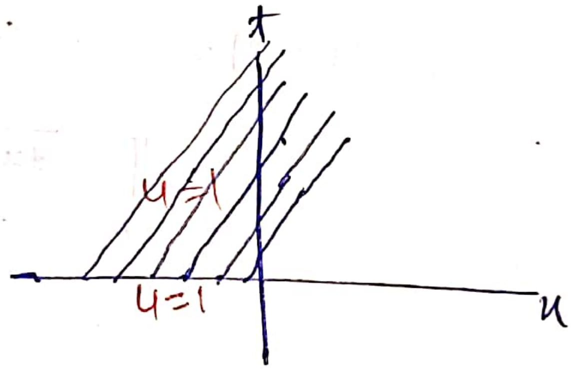
$$\frac{dx}{dt} = u \quad \text{and} \quad u_t + u u_x = \frac{du}{dt} = 0$$

for a perticular char. line passing through $(x_0, 0)$, $x_0 \le 0$

(1)
$$\left.\frac{dx}{dt}\right|_{(x_0,0)} = u(x_0, 0) = 1 \quad \text{if } x_0 \le 0$$

$\Rightarrow$ $x - x_0 = t$.

$\Rightarrow$ $\boxed{x = t + x_0}$, $x_0 \le 0$




represents char. line with slope 1 for $x \le 0$

Now, to find the value of $u(x,t)$ at char. lines enough to find at initial point because it is constant over a perticular char. line

So, And given $u(x,0) = 1$ if $x < 0$

$\Rightarrow u(x,t) = 1$ if $x < 0$ — (1)

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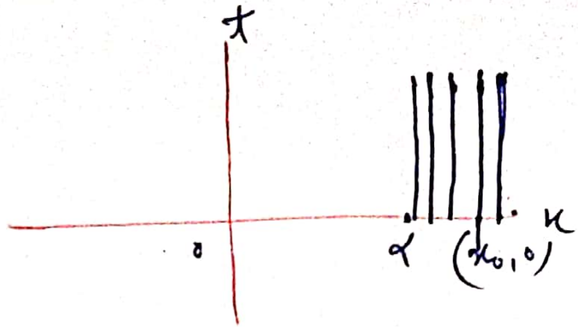
② step $\left.\frac{dx}{dt}\right|_{(x_0,0)} = u(x_0,0) = 0$ if $x > \alpha$

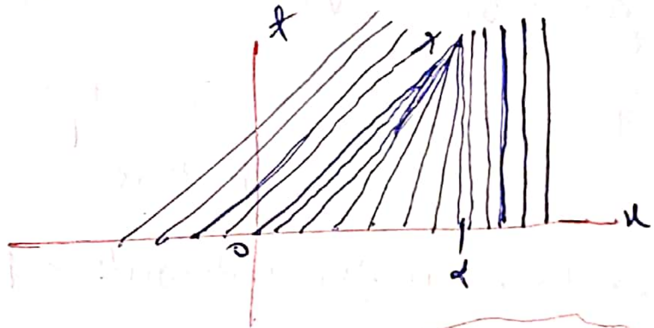



$\Rightarrow \quad x - x_0 = 0 \Rightarrow x = x_0$

Since initially $u = 0$ so,

$u(x,t) = 0$ if $x > \alpha$ — ②

③ $\left.\frac{dx}{dt}\right|_{(x_0,0)} = u(x_0,0) = 1 - \frac{x_0}{\alpha}$




$\Rightarrow \quad x - x_0 = \left(1 - \frac{x_0}{\alpha}\right)t$

$\Rightarrow \quad x = t - \frac{x_0 t}{\alpha} + x_0$

$\Rightarrow \quad \boxed{x = t\left(1 - \frac{x_0}{\alpha}\right) + x_0}$ — ⊛

$\Rightarrow \quad x - t = x_0\left(1 - \frac{t}{\alpha}\right)$

$\Rightarrow \quad x_0 = \frac{\alpha(x-t)}{(\alpha - t)}$

$1 - \frac{x_0}{\alpha} = 1 - \frac{x-t}{\alpha - t}$

$= \frac{\alpha - t - x + t}{\alpha - t}$

$= \frac{\alpha - x}{\alpha - t}$

represents char. lines with slope $1 - \frac{x_0}{\alpha}$

and $1 - \frac{x_0}{\alpha} \to 1$ as $x_0 \to 0$

and, $1 - \frac{x_0}{\alpha} \to 0$ as $x_0 \to \alpha$

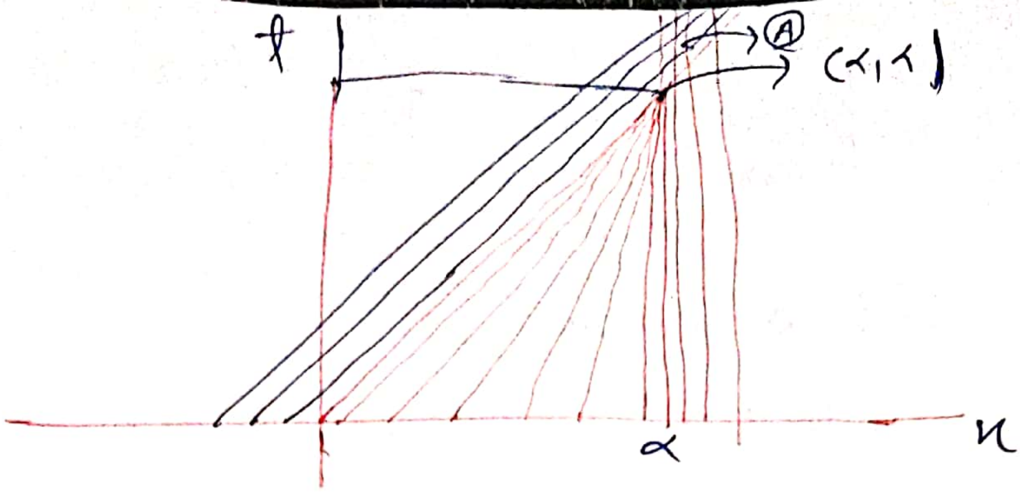
So, for every char. line in btw 0 and α initially $u = 1 - \frac{x_0}{\alpha}$ and remain same over the line

we can calculate the value of $x_0$ from ⊛ for given $(x,t)$

$\Rightarrow \quad u(x,t) = \frac{x - \alpha}{t - \alpha}$ for $t < x \le \alpha$

combining ① ② & ③

$$u(x,t) = \begin{cases} 1, & x \le t \\ \frac{(x-\alpha)}{(t-\alpha)}, & t < x \le \alpha \\ 0, & x > \alpha \end{cases}$$




Since we are getting a region where characteristic lines are intersecting each other, so at every point of Region A we get multiple values of $u(x,t)$ due to intersection of char. lines and $u(x,t)$ is decreasing from 1 to 0, so, Shock waves come into the picture.

→ $(\alpha,\alpha)$ is the intersection pt of two char. lines so,

$$\frac{dx}{dt} = s = \frac{u_l + u_r}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{1}{2}t + c \quad (\text{Passing through } (\alpha,\alpha)) \text{ so}$$

$$\alpha = \frac{1}{2}\alpha + c \Rightarrow \alpha = \frac{1}{2}\alpha$$

$$\Rightarrow \quad \boxed{x = \frac{1}{2}(t+\alpha)}$$

So, there is a discontinuous solution

$$u(x,t) = \begin{cases} 1 & \text{if } x \le \frac{(t+\alpha)}{2} \\ 0 & \text{if } x > \frac{(t+\alpha)}{2} \end{cases}$$