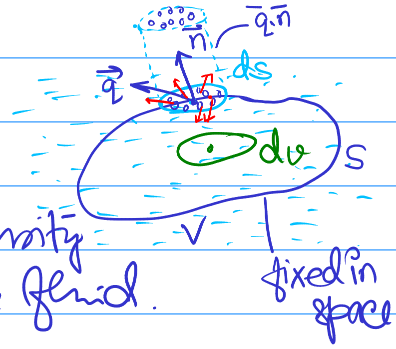
Equation of Continuity (Control volume approach)



Let, we consider a control volume V of a fluid enclosed by surface S fixed in space. Let the density of the fluid is ρ and $\bar{q}$ be the velocity of the fluid. Then, by principle of conservation of mass:

The rate of increase of mass of fluid within the volume V = The net influx of the mass across the surface S into the volume V per unit time

The mass of fluid within volume element $dv = \rho\, dv$

Total mass of fluid within volume $V = \int_V \rho\, dv$

The local rate of increase of mass within volume $V = \frac{\partial}{\partial t}\int_V \rho\, dv$

$$= \int_V \frac{\partial \rho}{\partial t}\, dv$$

Let $\bar{q}$ be the fluid velocity at any point of surface element ds of surface S and let $\bar{n}$ be the outward unit normal vector to ds.

Then,

the mass of fluid crossing area ds in unit time from inside to outside (outflux) = density × normal velocity × time × surface area

$$= \rho \times \bar{q}\cdot\bar{n} \times 1 \times ds$$

$$= \rho\,\bar{q}\cdot\bar{n}\, ds$$

$\therefore$ Total outflux of mass across surface S per unit time

$$= \int_S \rho\, \bar{q}\cdot\bar{n}\, ds$$

$\therefore$ Total influx of the mass within volume V across surface S per unit time

$$= -\int_S \rho\, \bar{q}\cdot\bar{n}\, ds$$

$$= -\int_V \nabla\cdot(\rho\bar{q})\, dv \qquad \text{(By Gauss div theorem)}$$

So, by principle of conservation of mass,

$$\int_V \frac{\partial \rho}{\partial t}\, dv = -\int_S \rho\vec{q}\cdot\vec{n}\, ds$$

(Integral form of eqn of continuity)

local rate of increase of mass within V

net influx of mass within V across S

If $\rho\vec{q}$ is continuously differentiable over volume V, then

$$\int_V \frac{\partial \rho}{\partial t}\, dv = -\int_V \nabla\cdot(\rho\vec{q})\, dv \qquad \text{(by G.D.T)}$$

$$\Rightarrow \int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q}) \right\} dv = 0$$

Since V is arbitrary, and conservation of mass holds throughout the flow-field, so

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q}) = 0}$$

Differential form of eqn of continuity.

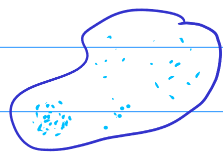
or,

$$\boxed{\frac{d\rho}{dt} + \rho\nabla\cdot\vec{q} = 0}$$

Particular forms of eqn of continuity -

① For steady flow, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \boxed{\nabla\cdot(\rho\vec{q}) = 0}$

② For homogeneous and incompressible fluid,



the density ρ is constant throughout flow field,

$$\boxed{\nabla\cdot\vec{q} = 0}$$

If fluid motion is of potential kind, $\vec{q} = -\nabla\phi$

$$\nabla\cdot\vec{q}=0 \quad \& \quad \vec{q}=-\nabla\phi$$

$$\Rightarrow \quad \nabla\cdot(-\nabla\phi)=0$$

$$\Rightarrow \quad \boxed{\nabla^2\phi=0}$$ Laplace eqn in velocity potential ϕ.

eqn of continuity for irrotational motion of incompressible fluid

③ A fluid motion is possible only when eqn of continuity is satisfied.

④ Eqn of Continuity, when $\vec{q}=q_\lambda\hat{a}_1+q_\mu\hat{a}_2+q_\nu\hat{a}_3$ (λ,μ,ν),

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{q})=0, \quad \vec{F}=\rho\vec{q}=\rho q_\lambda\hat{a}_1+\rho q_\mu\hat{a}_2+\rho q_\nu\hat{a}_3$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\lambda}(h_2h_3\rho q_\lambda)+\frac{\partial}{\partial\mu}(h_3h_1\rho q_\mu)+\frac{\partial}{\partial\nu}(h_1h_2\rho q_\nu)\right]=0$$

(i) in (x,y,z) co-ordinates, $h_1=1$, $h_2=1$, $h_3=1$, $\vec{q}=u\hat{i}+v\hat{j}+w\hat{k}$

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

or $$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

(ii) in (R,θ,z) co-ordinate, $h_1=1$, $h_2=R$, $h_3=1$,

$$\frac{\partial\rho}{\partial t}+\frac{1}{R}\left[\frac{\partial}{\partial R}(R\rho q_R)+\frac{\partial}{\partial\theta}(\rho q_\theta)+\frac{\partial}{\partial z}(R\rho q_z)\right]=0$$

(iii) in (r,θ,ψ) co-ordinate

⑤ One-dimensional motion.

(a) (x,y,z) co-ordinates; $\rho = \rho(x,t)$ $\vec{q} = u(x,t)\hat{i} + v(x,t)\hat{j} + w(x,t)\hat{k}$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

(b) (R,θ,z) co-ordinate : Cylindrically symmetric motion or flow.

(c) (r,θ,ψ) co-ordinate: Spherically symmetric flow.

⑥ Derive eq^n of continuity in (x,y,z), (R,θ,z) and (r,θ,ψ) co-ordinate from first principle (conservation of mass).

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