WEAKLY CONVERGENT

A sequence $\langle x_n \rangle$ in $(X, \|\|)$ is said to be weakly convergent at x if $\forall f \in X'$ we have

$\lim_{n\to\infty} f(x_n) \Rightarrow f(x)$ i.e $x_n \xrightarrow{w} x$ (Notation)

Theorem 1 Let X be a reflexive space. Then any bounded sequence contains a weakly convergent subsequence.

Theorem 2 Let M be a closed convex set in a norm space in norm topology then it is also weakly closed too.

Theorem If $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences in Hilbert space H s.t $x_n \xrightarrow{w} x$ & $y_n \xrightarrow{w} y$

Then $\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$

Proposition Let M be a bounded closed & convex in a (real) Hilbert space H. Let F be a nonexpansive map from M to M. Then There is a fixed Point of F in M. Moreover if

$x_0 \in M$ $\quad x_n = F(x_{n-1})$ and $y_n = \frac{1}{n} \sum_{k=0}^{n-1} x_k$

Then the sequence $\langle y_n \rangle_{n=1}^{\infty}$ converges weakly to the fixed point.

Proof $y_n \in M$ Therefore $\langle y_n \rangle$ is a bounded sequence in a Hilbert space H ~~and hence~~ which is reflexive. Thus $\exists$ a subsequence $\langle y_{n_k} \rangle$ of y_n s.t $y_{n_k} \xrightarrow{w} \tilde{x}$

and $\tilde{x} \in M$ [as M is bounded closed convex using Theorem 2 ~~and~~]

Fix such y_{n_k}

Claim $\lim_{n\to\infty} \|Fy_n - y_n\| = 0$

consider for any $k \in \mathbb{N}$

$$\|F^k(x_0) - y_n\|^2 = \|F^k(x_0) - F(y_n) + F(y_n) - y_n\|^2$$

$$= \|F^k(x_0) - F(y_n)\|^2 + \|F(y_n) - y_n\|^2$$

$$+ 2\langle F^k(x_0) - F(y_n), F(y_n) - y_n \rangle$$

Takeing sum from $k = 0$ to $n-1$ and then dividing by n we get

$$\frac{1}{n}\sum_{k=0}^{n-1} \|F^k(x_0) - y_n\|^2 = \frac{1}{n}\sum_{k=0}^{n-1} \|F^k(x_0) - F(y_n)\|^2 + \|F(y_n) - y_n\|^2$$

$$+ 2\langle y_n - F(y_n), F(y_n) - y_n \rangle$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \|F^k(x_0) - F(y_n)\|^2$$

$$- \|F(y_n) - y_n\|^2$$

Thus we get

$$\|F(y_n) - y_n\|^2 = \frac{1}{n}\sum_{k=0}^{n-1} \|F^k(x_0) - F(y_n)\|^2 - \frac{1}{n}\sum_{k=0}^{n-1}\|F^k(x_0) - y_n\|^2 \quad (*)$$

Applying non expansiveness on $\|F^k(x_0) - F(y_n)\| \leq \|F^{k-1}(x_0) - y_n\|$

As the sum in $(*)$ runs from $k=0$ to $n-1$ we will take out to avoid negative in power 1st term and then apply non expansiveness

$$\|F(y_n) - y_n\|^2 \leq \frac{1}{n}\|x_0 - F(y_n)\|^2 + \frac{1}{n}\sum_{k=1}^{n-1}\|F^{k-1}(x_0) - y_n\|^2 - \frac{1}{n}\sum_{k=0}^{n-1}\|F^k(x_0) - y_n\|^2$$

$$\|F(y_n) - y_n\|^2 \leq \frac{1}{n}\|x_0 - F(y_n)\|^2 - \frac{1}{n}\|F^{n-1}(x_0) - y_n\|^2$$

as $x_0 - F(y_n)$ and $F^{n-1}(x_0) - y_n$ are bounded sequence as M is bounded Therefore Takeing lim $n \to \infty$ we get

$$\|F(y_n) - y_n\| \to 0$$

<u>Claim</u> To show $\hat{x}$ is the fixed point of F

i.e $F(\hat{x}) = \hat{x}$ equivalent to show

$$\|F(\hat{x}) - \hat{x}\|^2 \leq 0$$

i.e To show $\langle F(\hat{x}) - \hat{x}, F(\hat{x}) - \hat{x}\rangle \leq 0$

Consider for any $z \in H$ we will show that

$\langle z - f(z) + f(y_{n_k}) - y_{n_k}, z - y_{n_k} \rangle \geq 0$ for any $k \in \mathbb{N}$

$= \langle z - y_{n_k}, z - y_{n_k} \rangle - \langle f(z) - f(y_{n_k}), z - y_{n_k} \rangle$

$\geq \|z - y_{n_k}\|^2 - \|z - y_{n_k}\|^2 = 0$

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because $\langle f(z) - f(y_{n_k}), z - y_{n_k} \rangle \leq \|f(z) - f(y_{n_k})\| \cdot \|z - y_{n_k}\|$

by using Cauchy Schwartz Inequality

Now further we have

$\langle f(z) - f(y_{n_k}), z - y_{n_k} \rangle \leq \|z - y_{n_k}\|^2$

Using non expansiveness and then mod

$-\langle f(z) - f(y_{n_k}), z - y_{n_k} \rangle \geq -\|z - y_{n_k}\|^2$

Now we have $z - y_{n_k} \xrightarrow{w} z - \hat{u}$

and $z - f(z) + f(y_{n_k}) - y_{n_k} \xrightarrow{w} z - f(z)$

as $\|f y_n - y_n\| \to 0$

Thus we have using Theorem 3 that

$\langle z - f(z), z - \hat{u} \rangle \geq 0$ (**) $\forall z \in H$

we choose $z = (1-t)\hat{u} + t f(\hat{u})$ $t \in (0,1)$

Clearly $z \in M$ being a convex set

putting value of z in (**) we get

$$\langle (1-t)\hat{x} + tF(\hat{x}) - F((1-t)\hat{x} + tF(\hat{x}), -t\hat{x} + F(\hat{x}) \rangle \geq 0$$

Dividing by t and taking $t \to 0$ and using the fact that inner product is a continous function and F is a continous function on M we get

$$\langle \hat{x} - F(\hat{x}), -\hat{x} + F(\hat{x}) \rangle \geq 0$$

$$\Rightarrow \quad \|\hat{x} - F(\hat{x})\|^2 \leq 0$$

<u>Claim</u> $\{y_n\} \xrightarrow{w} \hat{x}$

Let $\{y_{n_\ell}\}$ be another subsequence of y_n s.t $\{y_{n_\ell}\} \xrightarrow{w} v$ our aim is to show that $v = \hat{x}$

Note 'v' is a fixed too we can prove similarly like $\hat{x}$

Now first observe that if x is any fixed point of f then

$$\|x_n - x\|^2 = \|f(x_{n-1}) - f(x)\|^2 \leq \|x_{n-1} - x\|^2 \quad \text{(A)}$$

Let $a_n = \|x_n - x\|^2$ Then $a_n \leq a_{n-1}$ hence $\{a_n\}$ is a decreasing sequence also $a_n \geq 0$

Thus lower bounded and hence ~~TO~~ $\{a_n\}$ sequence converges to it's infinium

Let $\phi(x) = \lim\limits_{n\to\infty} \|x_n - x\|^2 = \inf\limits_{n \in \mathbb{N}} \|x_n - x\|^2$

as $\hat{x}$ is a fixed point we have

$$\varphi(\hat{x}) = \inf_{n \in \mathbb{N}} \|x_n - x\|^2$$

Thus $\varphi(\hat{x}) \leq \|x_k - x\|^2 = \|x_k - v + v - \hat{x}\|^2$

$$\varphi(\hat{x}) \leq \|x_k - v\|^2 + \|v - \hat{x}\|^2 + 2\langle x_k - v, v - \hat{x}\rangle$$

Now Taking sum from $k=0$ to $n-1$ and dividing by n we get that

$$\varphi(\hat{x}) \leq \frac{1}{n}\sum_{k=0}^{n-1} \|x_k - v\|^2 + \|v - \hat{x}\|^2 + 2\langle y_n - v, v - \hat{x}\rangle$$

Let $n = n_k$ and Taking $k \to \infty$ we get

$$\varphi(\hat{x}) \leq \varphi(v) - \|\hat{x} - v\|^2 \quad \text{— (B)}$$

because $\lim_{k\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \|x_k - v\|^2 = \lim_{k\to\infty} \|x_k - v\|^2 = \varphi(v)$ [v is a fixed Point too]

Using Cauchy Ist Theorem on limits

also $\lim_{k\to\infty} \frac{1}{n_k}\sum_{k=0}^{n_k-1} \|x_k - v\|^2 = \lim_{k\to\infty} \|x_k - v\|^2$

as subsequential limit and sequential limit are same for convergent sequence.

also $y_{n_k} \xrightarrow{w} \hat{x}$

In a similar way we can prove that

$$\varphi(v) \leq \varphi(\hat{x}) - \|\hat{x} - v\|^2 \quad \text{— (C)}$$

from (B) and (C) we have

$$\|\hat{x} - v\|^2 \leq 0 \Rightarrow \boxed{\hat{x} = v}$$

Hence we have shown that any subsequential weakly limit of y_n is $\hat{x}$.

we will claim using this that $y_n \xrightarrow{w} \hat{x}$

suppose that is not True ~~Then~~ i.e

$y_n \not\xrightarrow{w} \hat{x}$ Then $\exists$ $T_1 \in H^*$ such that

$T_1(y_n) \not\to \hat{x}$ Thus Then $\exists$ a subsequence

y_{n_m} say s.t

$|T(y_{n_m}) - T(\hat{x})| \geq \varepsilon_0$ — D $\forall$ $m \in \mathbb{N}$

~~for~~ some $\varepsilon_0 > 0$

Now $\langle y_{n_m} \rangle$ is again a bounded sequence in a Hilbert (reflexive) space which will have a convergent subsequence weakly but This will contradict D. Therefore our assumption that $y_n \not\xrightarrow{w} \hat{x}$ is wrong.