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Q8: Let F be a contraction defined on a metric space that is not assumed to be complete
Prove that
$$\inf_x d(x, Fx) = 0$$

Sol^n since F is a contraction $F: X \longrightarrow X$

$\therefore d(Fx, Fy) \leq \theta\, d(x,y)$ $\quad \theta \in [0,1)$ $\forall x, y \in X$

claim

$\inf_x d(x, Fx) = 0$ even if X is not complete

Fix $x_0 \in X$
then take a sequence
$\langle x_0, Fx_0, F^2x_0 \dots \rangle$
$= \langle x_0, x_1, x_2, \dots \rangle$
So $x_n = F^n x_0 \Rightarrow x_{n+1} = F(x_n)$
Now to show $d(x_i, Fx_i) \longrightarrow 0$ for $i \longrightarrow \infty$

$\Rightarrow \inf_x d(x, Fx) = 0$

~~If let $d(x, Fx)$~~ $\because$ let $\inf_x d(x, Fx) = l$

let $l \geq 0$
If $l \neq 0 \Rightarrow l > 0$

so by Archimedian property $\exists\, n \in \mathbb{N}$

$0 < \frac{1}{n} < l$

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which a contradiction since l is our infimum

$\therefore \inf d(x, Fx) = 0$

If $d(x_0, Fx_0) = 0$ then our claim holds

let $d(x_0, Fx_0) \neq 0$

$\Rightarrow d(x_0, x_1) \neq 0$

Now $d(x_n, x_{n-1}) = d(Fx_{n-1}, x_{n-1})$

$d(Fx_{n-1}, Fx_{n-2}) \leq \theta\, d(x_{n-1}, x_{n-2})$

$\vdots$

$\leq \theta^{n-1} d(x_1, x_0)$

taking limit

$\lim_{n\to\infty} d(Fx_{n-1}, x_{n-1}) \leq \lim_{n\to\infty} \theta^{n-1} d(x_1, x_0)$

$\because \theta \in [0, 1)$

$\therefore \lim_{n\to\infty} \theta^{n-1} = 0$

$\therefore d(Fx_{n-1}, x_{n-1}) \to 0$ as $n \to \infty$

$\boxed{\therefore \inf d(x, Fx) = 0}$

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Q11 Give an example of a discontinuous map $f:\mathbb{R}\to\mathbb{R}$ s.t $f \circ f$ is a contraction find the fixed point of F.

Sol^n
$$f(x) = \begin{cases} 1/2 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

$$f \circ f(x) = \begin{cases} 1/2 & \forall x \end{cases}$$

$\therefore |f_x - f_y| = 0 \leq \theta|x-y| \quad \theta \in [0,1)$
$\forall x, y \in \mathbb{R}$

$\therefore f \circ f(x)$ is a contraction

using the theorem : if f be a mapping of a complete metric space into itself, s.t for some $m \in \mathbb{N}$, f^m is contractive. Then f has a unique fixed point point. It is the limit of every seq $\{F_x^k\}$, for arbitrary $x \in X$

$\therefore$ $\{f_x^k\} = 1/2 \quad \forall k \geq 2 \quad \therefore \lim_{k\to\infty} \{f_x^k\} = 1/2$

$\therefore$ f has a unique fixed point.

fixed point of $f = 1/2$

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12 Extend the theorem in Problem 7 by showing that the fixed point is the limit of $f^n x$ for arbitrary x.

Problem 7: If X is a compact metric space and f is a mapping from X to X s.t

$$d(f_x, f_y) < d(x,y) \text{ when } x \neq y, \text{ then } f$$
has a unique fixed point.

Sol: Let a is the unique fixed point for arb x then to show

$$f^n(x) \longrightarrow a$$
$$x_0 = x$$

then $x_{n+1} = f(x_n)$

Case I Suppose if for some $k \in \mathbb{N}$

$$f^k(x) = a$$
$$\Rightarrow x_{k+1} = a$$

then $x_{k+2} = f(x_{k+1}) = a$

$$\Rightarrow x_n \longrightarrow a$$

Case 2 Let $f(x_n) \neq a \quad \forall n \in \mathbb{N}$

It means we can take
$$x_n \neq a$$

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$\Rightarrow d(x_n, a) > 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow d(x_{n+1}, a) = d(f(x_n), f(a))$

$< d(x_n, a)$

It means $d(x_n, a)$ is a decreasing positive sequence, it is cgt

So let $l = \lim_{n \to \infty} d(x_n, a)$

where $l \geq 0$

if $l = 0 \Rightarrow x_n \to a$ so let $l > 0$

As X is compact So $\exists$ subsequence $x_{n_j} \to y$

as f is continuous

so $f(x_{n_i}) \to f(y)$ as $i \to \infty$

$\Rightarrow x_{n_i+1} \longrightarrow f(y)$ as $i \to \infty$

Since $d(x_n, a) \to l$ as $n \to \infty$

$d(x_{n_i}, a) \to l$, $d(x_{n_i+1}, a) \to l$

as these are subsequence of $d(x_n, a)$

By continuity of metric

$d(x_{n_i}, a) \longrightarrow d(y, a)$

$d(x_{n_i+1}, a) = d(f(x_{n_i}), f(a))$

$\to d(f(y), a)$ — $(*)$

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$d(y,a) = l = d(f(y), a)$

$= d(f(y), f(a))$

If $y \neq a$ then $d(f(y), f(a)) < d(y,a)$

But this contradiction

So $y = a \Rightarrow l = d(y,a) = 0$

this shows that $d(x_n, a) \longrightarrow 0$

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