# R-07 R-07<br>Convex and Nonsmooth Analysis

#### Closed Convex Hull of a Set

**Closed Convex Hull of a Set**<br>Closed convex hull of a set  $S \subseteq \mathbb{R}^n$ , denoted by  $\overline{co}S$ , is the in<br>closed convex sets containing S.<br>Theorem The closed convex hull  $\overline{co}S$  is the closure of the conve **nvex Hull of a Set**<br>, denoted by  $\overline{\cos}$ , is the intersection of all<br>, is the closure of the convex hull of S, that

**Closed Convex Hull of a Set**<br>Closed convex hull of a set  $S \subseteq \mathbb{R}^n$ , denoted by  $\overline{cos}$ , is the intersection of all<br>closed convex sets containing  $S$ .<br>Theorem The closed convex hull  $\overline{cos}$  is the closure of the con is,  $\overline{\cos}$  = cl(coS).

*Example*  $cl(cos) \neq co(cls)$ . Let  $S = \{(0,0)\} \cup \{(t,1): t \geq 0\}$ .



*Theorem* If S is a set in  $\mathbb{R}^n$  then

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**Proof**  $S \subseteq \text{co}S \implies \text{cl}S \subseteq \text{cl}(\text{co}S) \implies \text{co}(\text{cl}S) \subseteq \text{co}(\text{cl}(\text{co}S)) = \text{cl}(\text{co}S)$ 

# **Boundedness and compactness of convex hull**<br>() If S is bounded then coS is bounded. **Boundedness and compactness of convex hull**<br>Theorem i) If S is bounded then coS is bounded.<br>ii) If S is compact then coS is compact.<br>Proof i) If S is bounded there exists  $M > 0$  such that **Boundedness and compactness of convex**<br>Theorem i) If S is bounded then coS is bounded.<br>ii) If S is compact then coS is compact.<br>Proof i) If S is bounded there exists  $M > 0$  such that<br> $||u|| \le M$ ,  $\forall u \in S$ .

$$
||u|| \le M, \qquad \forall u \in S.
$$

**Boundedness and compactness of convex hull<br>
Theorem i) If S is bounded then coS is bounded.**<br>
ii) If S is compact then coS is compact.<br>
Proof i) If S is bounded there exists  $M > 0$  such that<br>  $||u|| \le M$ ,  $\forall u \in S$ .<br>
Let  $x \$ **Boundedness and compactness of convex hull**<br>
Theorem i) If *S* is bounded then co*S* is bounded.<br>
ii) If *S* is compact then co*S* is compact.<br> *Proof* i) If *S* is bounded there exists  $M > 0$  such that<br>  $||u|| \le M$ ,  $\forall u \in$ **Boundedness and compactne**<br>heorem i) If *S* is bounded then co*S* is bounded.<br>
If *S* is compact then co*S* is compact.<br> *roof* i) If *S* is bounded there exists  $M > 0$  such tha<br>  $||u|| \le M$ ,  $\forall u \in$ <br>
et  $x \in \text{coS}$ . Then b  $\Delta_{n+1}$  such that  $x = \sum_{i=1}^{n+1} \alpha_i x_i$ . Now, **Iness and compactness of convex h**<br>ded then co*S* is bounded.<br>co*S* is compact.<br>I there exists  $M > 0$  such that<br> $||u|| \le M$ ,  $\forall u \in S$ .<br>Carathéodory theorem there exist  $x_1, x_2, ..., x_n$ <br> $\sum_{i=1}^{n+1} a_i x_i$ . Now,<br> $\sum_{i=1}^k a_i x_i || \$ 

$$
||x|| = ||\sum_{i=1}^{k} \alpha_i x_i|| \le \sum_{i=1}^{k} \alpha_i ||x_i|| \le (\sum_{i=1}^{k} \alpha_i) M = M.
$$

Hence  $\cos$  is bounded.

Theorem i) If S is bounded then coS is bounded.<br>
ii) If S is compact then coS is compact.<br>
Proof i) If S is bounded there exists  $M > 0$  such that<br>  $||u|| \le M$ ,  $\forall u \in S$ .<br>
Let  $x \in \cos S$ . Then by Carathéodory theorem there exis  $x_2, ..., x_{n+1} \in S, \alpha \in$ <br>  $M = M.$ <br>
be a sequence in co*S*<br>
E *S* and ii) If *S* is compact then co*S* is compact.<br> *Proof i*) If *S* is bounded there exists *M* > 0 suo<br>  $||u|| \le M$ ,<br>
Let  $x \in \cos S$ . Then by Carathéodory theoren<br>  $\Delta_{n+1}$  such that  $x = \sum_{i=1}^{n+1} \alpha_i x_i$ . Now,<br>  $||x|| = ||\sum_{i=1}^{k}$ between the costal solution.<br>
Alternatives and that<br>  $||u|| \leq M$ ,  $\forall u \in S$ .<br>  $\forall u \in S$ .<br>  $\forall u \in S$ .<br>  $\sum_{i=1}^{n+1} \alpha_i x_i$ . Now,<br>  $= \left\| \sum_{i=1}^{k} \alpha_i x_i \right\| \leq \sum_{i=1}^{k} \alpha_i ||x_i|| \leq (\sum_{i=1}^{k} \alpha_i) M = M$ .<br>
Alternative cost is closed.  $k \gamma k \gamma k \in \mathcal{S}$  and  $_2$  , …,  $x_{n+1}$   $\in$  3 driu  $x_2^k$ , ...,  $x_{n+1}^k \in S$  and  $k = (\alpha^k, \alpha^k, \alpha^k) \in$ 1,  $a_2$ , ...,  $a_{n+1}$ )  $\in \Delta_{n+1}$  $k \alpha^k \alpha^k \in \mathbb{R}$  $_{2}$ , ...,  $a_{n+1}$ )  $\in$   $\Delta_{n+1}$  suc  $\alpha_{n+1}^{k}$  , ...,  $\alpha_{n+1}^{k}) \in \Delta_{n+1}$  such that exists  $M > 0$  such that<br>  $||u|| \le M$ ,  $\forall u \in S$ .<br>
éodory theorem there exist  $x_1, x_2, ..., x_{n+1} \in S$ .<br>  $x_i$ . Now,<br>  $|x_i x_i|| \le \sum_{i=1}^k \alpha_i ||x_i|| \le (\sum_{i=1}^k \alpha_i) M = M$ .<br>
i closed. Let  $x \in cl(cos)$ . Let  $\{x^k\}$  be a sequence i<br>  $k$  we can c  $k = \nabla^{n+1} \alpha^k x^k$  $i \lambda i$ .  $k_{\gamma}k$  $i$   $\cdot$  $n+1 \overline{a_i^k x_i^k}.$ Let  $x \in \cos S$ . Then by Carathéodory theorem there exist  $x_1, x_2, ..., x_{n+1} \in S$ ,  $\alpha \in \Delta_{n+1}$  such that  $x = \sum_{i=1}^{n+1} \alpha_i x_i$ , Now,<br>  $||x|| = ||\sum_{i=1}^{k} \alpha_i x_i|| \le \sum_{i=1}^{k} \alpha_i ||x_i|| \le (\sum_{i=1}^{k} \alpha_i) M = M$ .<br>
Hence  $\cos S$  is bounded.<br>
(i  $|a_i x_i| \le \sum_{i=1}^k \alpha_i ||x_i|| \le (\sum_{i=1}^k \alpha_i) M = M.$ <br>
is closed. Let  $x \in \text{cl}(\cos S)$ . Let  $\{x^k\}$  be a sequence in  $\cos x^k$  we can choose  $x_1^k, x_2^k, ..., x_{n+1}^k \in S$  and<br>  $\Delta_{n+1}$  such that<br>  $x^k = \sum_{i=1}^{n+1} \alpha_i^k x_i^k$ .<br>
d S is  $\|x_i\| \le \sum_{i=1}^k \alpha_i \|x_i\| \le (\sum_{i=1}^k \alpha_i) M = M.$ <br>
closed. Let  $x \in \text{cl}(\cos)$ . Let  $\{x^k\}$  be a sequence in  $\cos k$ <br>  $\kappa$  we can choose  $x_1^k, x_2^k, ..., x_{n+1}^k \in S$  and<br>  $n+1$  such that<br>  $x^k = \sum_{i=1}^{n+1} \alpha_i^k x_i^k$ .<br>
S is comp and  $\alpha$  and

sequences  $\{\alpha^k\}$  and  $\{x^k_i\}$ ,  $i=1,2,...,n+1$  which converge. Let  $x^{k_l}_i \rightarrow x_i, i=1,2,...,n+1$  $i, i =$ 1,2, ...,  $n + 1$  and  $\alpha^{k_l} \rightarrow \alpha$ . As S and  $\Delta_{n+1}$  are closed it follows that  $x_i \in S$ ,  $\overline{\phantom{a}}$ 

$$
x = \sum_{i=1}^{n+1} \alpha_i x_i \in \text{coS}.
$$

# ?

**Cl(COS) = CO(ClS)?**<br>Theorem If S is compact then  $cl(cos) = co(clS)$ .<br>Proof As S is compact so  $coS$  is compact. Hence<br> $cl(cos) = cos$ .<br>As S is closed we have  $S = clS$ . Hence,

$$
cl(cos) = cos.
$$

$$
cl(cos) = cos = co(cls).
$$

. As is closed we have Hence, . Theorem If is bounded then . Proof As we have This implies . (1) Also we have From (1)-(3) we have

As  $S$  is bounded so  $cIS$  is compact. Hence by the previous theorem

$$
cl\big(co(clS)\big) = co(cl(clS)) = co(clS).
$$
 (2)

$$
co(clS) \subseteq cl(cos)
$$
 (3)

$$
\mathrm{cl}(\mathrm{co}S) \subseteq \mathrm{co}(\mathrm{cl}S) \subseteq \mathrm{cl}(\mathrm{co}S).
$$

#### Convex cone

**Convex c**<br>A set  $K \subseteq \mathbb{R}^n$  is said to be a cone if for  $x$ <br> $\alpha x \in K$ .<br>A convex cone  $K$  is a cone which is conve **CONVEX CONE**<br>
is said to be a cone if for  $x \in K$  and  $\alpha > 0$ , we have<br>  $\alpha x \in K$ .<br> *K* is a cone which is convex.

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A convex cone  $K$  is a cone which is convex.

**Convex cone**<br>  $A \text{ set } K \subseteq \mathbb{R}^n$  is said to be a cone if for  $x \in K$  and  $\alpha > 0$ , we have<br>  $\alpha x \in K$ .<br>
A convex cone K is a cone which is convex.<br>
Theorem A cone K is convex if and only if for every  $x, x' \in K$  we have<br>  $x +$  $x + x' \in K$ . A set  $K \subseteq \mathbb{R}^n$  is said to be a cone if for  $x \in K$  and  $\alpha > 0$ <br>  $\alpha x \in K$ .<br>
A convex cone K is a cone which is convex.<br>
Theorem A cone K is convex if and only if for every  $x$ ,  $x + x' \in K$ .<br>
Proof Let K be a convex cone. A convex cone *K* is a cone which is convex.<br>
Theorem A cone *K* is convex if and only if for every  $x, x' \in K$  we have<br>  $x + x' \in K$ .<br>
Proof Let *K* be a convex cone. Let  $x, x' \in K$ . As *K* is convex we have<br>  $\frac{1}{2}x + \frac{1}{2}$ 

*Proof* Let K be a convex cone. Let  $x, x' \in K$ . As K is convex we have

$$
\frac{1}{2}x + \frac{1}{2}x' \in K.
$$

$$
x + x' = 2\left(\frac{1}{2}x + \frac{1}{2}x'\right) \in K.
$$

$$
\lambda x \in K
$$
, and  $(1 - \lambda)x' \in K$ .

Hence by the assumption we have

$$
\lambda x + (1 - \lambda) x' \in K.
$$

#### Conical Combinations and Conical Hull

**Conical Combinations and Conical Hull**<br>Let  $\{x_i\}_{i=1}^k$  be a finite set of points in  $\mathbb{R}^n$ . A conical<br>combination of these points is a point of the form<br> $x = \sum_{i=1}^k \lambda_i x_i, \lambda_i \ge 0, i = 1, 2, ... k.$ **Conical Combinations and Conical Hull**<br>Let  $\{x_i\}_{i=1}^k$  be a finite set of points in  $\mathbb{R}^n$ . A conical<br>combination of these points is a point of the form<br> $x = \sum_{i=1}^k \lambda_i x_i, \lambda_i \ge 0, i = 1, 2, ... k$ .<br>Conical hull of a set **Conical Combinations and Conical Hull**<br>
Let  $\{x_i\}_{i=1}^k$  be a finite set of points in  $\mathbb{R}^n$ . A conical<br>
combination of these points is a point of the form<br>  $x = \sum_{i=1}^k \lambda_i x_i$ ,  $\lambda_i \ge 0$ ,  $i = 1, 2, ... k$ .<br>
Conical hul **Conical Combinations and Conical Hull**<br>
Let  $\{x_i\}_{i=1}^k$  be a finite set of points in  $\mathbb{R}^n$ . A conical<br>
combination of these points is a point of the form<br>  $x = \sum_{i=1}^k \lambda_i x_i$ ,  $\lambda_i \ge 0$ ,  $i = 1, 2, ... k$ .<br>
Conical hul Let  $\{x_i\}_{i=1}^k$  be a finite set of points in  $\mathbb{R}^n$ . A conical<br>combination of these points is a point of the form<br> $x = \sum_{i=1}^k \lambda_i x_i, \lambda_i \ge 0, i = 1, 2, \dots k$ .<br>Conical hull of a set  $S \subseteq \mathbb{R}^n$ , denoted by cone*S*, is

$$
x = \sum_{i=1}^{k} \lambda_i x_i, \ \lambda_i \ge 0, i = 1, 2, \dots k.
$$

combination of these points is a point of the form<br>  $x = \sum_{i=1}^{k} \lambda_i x_i, \lambda_i \ge 0, i = 1, 2, ... k.$ <br>
Conical hull of a set  $S \subseteq \mathbb{R}^n$ , denoted by cone*S*, is the set of<br>
all conical combinations of finite set of points of *S*.  $x = \sum_{i=1}^{k} \lambda_i x_i, \lambda_i \ge 0, i = 1, 2, ... k.$ <br>Conical hull of a set  $S \subseteq \mathbb{R}^n$ , denoted by cone*S*, is the set all conical combinations of finite set of points of *S*.<br>Clearly,  $0 \in \text{cone}S$ .<br>Theorem Conical hull of a set *S*

### Conical Convex Hull of a Set



#### Conical hull in terms of convex hull

**Conical hull in terms of convex hull**<br>
Theorem If S is a set in  $\mathbb{R}^n$  then cone  $S = \mathbb{R}^+(\cos)$ .<br>
Proof Let  $x \in \text{cone}S$ . Then there exist  $k \in \mathbb{N}$ ,  $x_i \in S$ ,  $\lambda_i \ge 0$ ,  $i = 1, 2, ..., k$ , such that<br>  $x = \sum_{i=1}^k \lambda_i x_i$ . **Conical hull in terms of con**<br>
If *S* is a set in  $\mathbb{R}^n$  then cone $S = \mathbb{R}^+$  (co.)<br>  $x \in \text{cone}S$ . Then there exist  $k \in \mathbb{R}$ <br>
such that<br>  $x = \sum_{i=1}^k \lambda_i x_i$ .<br>  $\frac{k}{n} \lambda_i$  If  $\hat{\lambda} = 0$  then  $x = 0 \in \mathbb{R}^+$  (co. S)

$$
x = \sum_{i=1}^k \lambda_i x_i.
$$

**Conical hull in terms of convex hull**<br>
Theorem If S is a set in  $\mathbb{R}^n$  then coneS =  $\mathbb{R}^+(cos)$ .<br>
Proof Let  $x \in \text{cone}S$ . Then there exist  $k \in \mathbb{N}$ ,  $x_i \in S$ ,  $\lambda_i \ge 0$ ,  $i = 1, 2, ..., k$ , such that<br>  $x = \sum_{i=1}^k \lambda_i x_i$ .  $x = \hat{\lambda} \sum_{i=1}^k \frac{\lambda_i}{\hat{\lambda}} x_i \in \mathbb{R}^+ (\cos).$ Theorem If S is a set in  $\mathbb{R}^n$  then coneS =  $\mathbb{R}^+(cos)$ .<br>
Proof Let  $x \in \text{coneS}$ . Then there exist  $k \in \mathbb{N}$ ,  $x_i \in S$ ,  $\lambda_i \ge 0$ ,  $i = 1, 2, ..., k$ , such that<br>  $x = \sum_{i=1}^k \lambda_i x_i$ .<br>
Let  $\hat{\lambda} = \sum_{i=1}^k \lambda_i$ . If  $\hat{\lambda} = 0$ there exist  $k \in \mathbb{N}$ ,  $x_i \in S$ ,  $\lambda_i \ge 0$ ,  $i =$ <br>  $\sum_{i=1}^{k} \lambda_i x_i$ .<br>  $x = 0 \in \mathbb{R}^+ (\cos)$ . Let  $\hat{\lambda} > 0$ . Then<br>  $\sum_{i=1}^{\lambda_i} \frac{\lambda_i}{\hat{\lambda}} x_i \in \mathbb{R}^+ (\cos)$ .<br>
Then there exist  $\alpha \ge 0$ ,  $x_i \in S$ ,  $\lambda_i \ge$ , such that<br>  $\sum_{i=1}$ 

$$
x = \alpha \sum_{i=1}^{k} \lambda_i x_i = \sum_{i=1}^{k} \alpha \lambda_i x_i \in \text{cone}S.
$$

Remark  $0 \in \mathbb{R}^+$  (coS) = coneS.

#### Conical hull in terms of convex hull

Theorem If S is a set in  $\mathbb{R}^n$  then cone $S = \mathbb{R}^+(\cos) = \cos(\mathbb{R}^+S)$ . **Conical hull in terms of convex hull**<br>
Theorem If S is a set in  $\mathbb{R}^n$  then cone $S = \mathbb{R}^+(\cos S) = \cos(\mathbb{R}^+ S)$ .<br>
Proof Let  $x \in \mathbb{R}^+(\cos S)$ . Then there exist  $\alpha \ge 0$ ,  $x_i \in S$ ,  $\lambda_i \ge 0$ ,  $i = 1, ..., n + 1$ ,  $\sum_{i=1}^k \lambda_i = 1$ **all hull in terms of convex hull<br>**  $\mathbb{R}^n$  **then cone** $S = \mathbb{R}^+(\cos) = \cos(\mathbb{R}^+S)$ **.<br>
Then there exist**  $\alpha \ge 0$ **,**  $x_i \in S$ **,**  $\lambda_i \ge$ **, such that<br>**  $\lambda_i x_i = \sum_{i=1}^k \lambda_i(\alpha x_i) \in \cos(\mathbb{R}^+S)$ **.<br>**  $\mathbb{R}^+S$ **). Then there exist u\_i \in \math Conical hull in terms of convex hull**<br>
Theorem If S is a set in  $\mathbb{R}^n$  then cone $S = \mathbb{R}^+(\cos S) = \cos(\mathbb{R}^+S)$ .<br> *Proof* Let  $x \in \mathbb{R}^+(\cos S)$ . Then there exist  $\alpha \ge 0$ ,  $x_i \in S$ ,  $\lambda_i \ge 0$ ,  $i = 1, ..., n + 1$ ,  $\sum_{i=1}^k \lambda_i =$ **all hull in terms of convex hull<br>**  $\mathbb{R}^n$  **then cone** $S = \mathbb{R}^+(cos) = co(\mathbb{R}^+S)$ **.<br>
Then there exist**  $\alpha \ge 0$ **,**  $x_i \in S$ **,**  $\lambda_i \ge$ **, such that<br>**  $\lambda_i x_i = \sum_{i=1}^k \lambda_i (\alpha x_i) \in co(\mathbb{R}^+S)$ **.<br>**  $\mathbb{R}^+S$ **). Then there exist u\_i \in \mathbb{** 

$$
x = \alpha \sum_{i=1}^{k} \lambda_i x_i = \sum_{i=1}^{k} \lambda_i (\alpha x_i) \in \text{co}(\mathbb{R}^+ S).
$$

$$
x=\sum_{i=1}^k\lambda_i u_i.
$$

As  $u_i \in \mathbb{R}^+$ S, there exist  $x_i \in S$ ,  $\alpha_i \geq 0$ ,  $i = 1, ..., n + 1$ , such that  $u_i = \alpha_i x_i$ . Hence

$$
x=\sum_{i=1}^k \lambda_i \alpha_i x_i.
$$

 $x = \alpha \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \lambda_i (\alpha x_i) \in \text{co}(\mathbb{R}^+ S).$ <br>Conversely, let  $x \in \text{co}(\mathbb{R}^+ S)$ . Then there exist  $u_i \in \mathbb{R}^+ S, \lambda_i \ge 0, i = 1, ..., n + 1, \sum_{i=1}^{k} \lambda_i = 1$ , such that<br> $x = \sum_{i=1}^{k} \lambda_i u_i$ .<br>As  $u_i \in \mathbb{R}^+ S$ , ther  $\frac{i^{ai}}{\hat{s}} x_i \in \mathbb{R}^+ (\cos).$ 

## cl(coneS) and cone(clS)

**cl(coneS) and cone(clS)**<br>We have  $cl(cos) = co(clS)$  if *S* is compact. But such a relation fails<br>to hold even when *S* is compact when "co" is replaced by "cone".<br>Example  $cl(coneS) \neq cone(clS)$ **cl(coneS) and cone(clS)**<br>We have cl(coS) = co(clS) if S is compact. But such a relation fails<br>to hold even when S is compact when "co" is replaced by "cone".<br>Example cl(coneS)  $\neq$  cone(clS)<br>Let  $S = \{(x, y): (x - 1)^2 + y^2 \le 1$ Example  $cl(coneS) \neq cone(cS)$ 

Let  $S = \{(x, y): (x - 1)^2 + y^2 \le 1\}.$ 



#### Closed conical hull

Closed conical hull of a nonempty set S in  $\mathbb{R}^n$ , denoted by  $\overline{\mathrm{cone}}S$ , is defined as  $\overline{\text{cone}}S$ :=  $\text{cl}(\text{cone}S)$ . **Closed conical hull**<br>Closed conical hull of a nonempty set S in  $\mathbb{R}^n$ , denoted by<br>  $\overline{\text{cone}}S := \text{cl}(\text{cone}S)$ .<br>
Theorem If S is a nonempty compact set in  $\mathbb{R}^n$  such that 0<br>  $\overline{\text{cone}} S = \text{cone} S$ .<br>
Proof We need to s

Theorem If S is a nonempty compact set in  $\mathbb{R}^n$  such that  $0 \notin \cos$  then

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**Closed conical hull**<br>
Closed conical hull of a nonempty set *S* in  $\mathbb{R}^n$ , denoted by cone*S*, is defined as<br>  $\overline{\text{cone}}S := \text{cl}(\text{cone}S)$ .<br>
Theorem If *S* is a nonempty compact set in  $\mathbb{R}^n$  such that  $0 \notin \text{coS}$  th over  $\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{$ **Closed conical hull**<br>
Closed conical hull of a nonempty set *S* in  $\mathbb{R}^n$ , denoted by come*S*, is defined as<br>  $\overline{\text{cone}}S := cl(\text{cone}S)$ .<br>
Theorem If *S* is a nonempty compact set in  $\mathbb{R}^n$  such that  $0 \notin \text{co}S$  then<br> **Closed conical hull of a nonempty set S in**  $\mathbb{R}^n$ **, denotes**  $\overline{\text{cone}} S := \text{cl}(\text{cone})$ **<br>
Theorem If S is a nonempty compact set in**  $\mathbb{R}^n$  **subsequence of**  $\overline{\text{cone}} S = \text{cone}$ **.<br>
Proof We need to show**  $\text{cl}(\text{cone} S) = \text{cone} S$ **.<br> Sed conical hull**<br>
t S in  $\mathbb{R}^n$ , denoted by cones, is defined as<br>  $\overline{e}S := cl(\text{cone}S)$ .<br>
t set in  $\mathbb{R}^n$  such that  $0 \notin \cos$  then<br>  $\overline{me} S = \text{cone} S$ .<br>
= cones.<br>
. We know  $\cos S = \mathbb{R}^+(\cos S)$ .<br>
closed. Let  $t_k x_k \in \mathbb{R}$ S:= cl(coneS).<br>
set in  $\mathbb{R}^n$  such that  $0 \notin \cos$  then<br>  $\overline{e} S = \text{cone} S$ .<br>
coneS.<br>
We know coneS =  $\mathbb{R}^+(\cos)$ .<br>
osed. Let  $t_k x_k \in \mathbb{R}^+(\cos)$  such that  $t_k x_k \rightarrow y$ <br>
As S is compact so  $\cos$  is compact. Thus then<br>  $x \neq 0$ 1 to show cone S is closed. We know cone  $S = \mathbb{R}^+(cos)$ .<br>
ugh to show  $\mathbb{R}^+(cos)$  is closed. Let  $t_k x_k \in \mathbb{R}^+(cos)$  such that  $t_k x_k \rightarrow y$ .<br>  $S$  and  $0 \notin cos$  so  $x_k \neq 0$ . As  $S$  is compact so  $cos$  is compact. Thus there<br>
sequ

$$
t_{k_l}||x_{k_l}|| \to ||y||.
$$

As  $||x_{k_l}|| \rightarrow ||x||$ ,  $||x_{k_l}|| \neq 0$ ,  $||x|| \neq 0$ 

$$
t_{k_l} = t_{k_l} \frac{\|x_{k_l}\|}{\|x_{k_l}\|} \to \frac{\|y\|}{\|x\|}.
$$

Let  $t = \frac{\|y\|}{\|x\|}$ . As  $x_{k_1} \to x$  and  $t_{k_1} \to x$  $||x||$   $\sim$   $\frac{E}{k_l}$   $\sim$   $\frac{E}{k_l}$ have  $y = \overline{t}x \in \mathbb{R}^+$  (coS).

Corollary If S is a nonempty compact set in  $\mathbb{R}^n$  such that  $0 \notin \cos$  then  $cl(coneS) = cone(clS).$