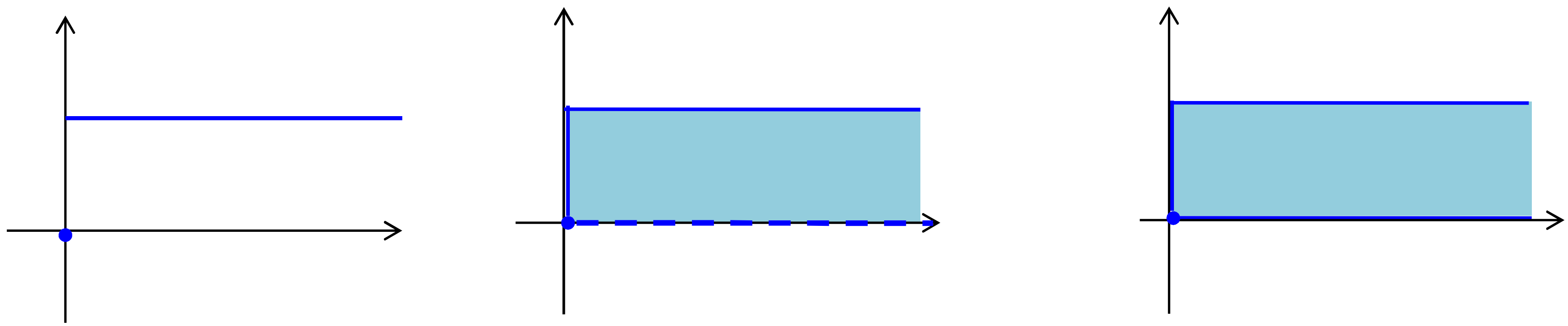
R-07
Convex and Nonsmooth Analysis

Closed Convex Hull of a Set

Closed convex hull of a set $S \subseteq \mathbb{R}^n$, denoted by $\overline{\text{co}}S$, is the intersection of all closed convex sets containing S.

Theorem The closed convex hull $\overline{\text{co}}S$ is the closure of the convex hull of S, that is, $\overline{\text{co}}S = \text{cl}(\text{co}S)$.

Example $\text{cl}(\text{co}S) \neq \text{co}(\text{cl}S)$. Let $S = \{(0,0)\} \cup \{(t,1): t \geq 0\}$.



$\boldsymbol{S = \textbf{cl}S}$ $\qquad$ $\boldsymbol{\textbf{co}S = \textbf{co}(\textbf{cl}S)}$ $\qquad$ $\boldsymbol{\textbf{cl}(\textbf{co}S)}$

Theorem If S is a set in $\mathbb{R}^n$ then

$$\text{co}(\text{cl}S) \subseteq \text{cl}(\text{co}S).$$

Proof $\quad S \subseteq \text{co}S \Longrightarrow \text{cl}S \subseteq \text{cl}(\text{co}S) \Longrightarrow \text{co}(\text{cl}S) \subseteq \text{co}\big(\text{cl}(\text{co}S)\big) = \text{cl}(\text{co}S)$

as $\text{cl}(\text{co}S)$ is a convex set.

Boundedness and compactness of convex hull

Theorem i) If S is bounded then $\mathrm{co}S$ is bounded.

ii) If S is compact then $\mathrm{co}S$ is compact.

Proof i) If S is bounded there exists $M > 0$ such that

$$\|u\| \leq M, \qquad \forall u \in S.$$

Let $x \in \mathrm{co}S$. Then by Carathéodory theorem there exist $x_1, x_2, \dots, x_{n+1} \in S, \alpha \in \Delta_{n+1}$ such that $x = \sum_{i=1}^{n+1} \alpha_i x_i$. Now,

$$\|x\| = \left\|\sum_{i=1}^{k} \alpha_i x_i\right\| \leq \sum_{i=1}^{k} \alpha_i \|x_i\| \leq \left(\sum_{i=1}^{k} \alpha_i\right) M = M.$$

Hence co$S$ is bounded.

ii) It is enough to show co$S$ is closed. Let $x \in \mathrm{cl}(\mathrm{co}S)$. Let $\{x^k\}$ be a sequence in co$S$ such that $x^k \to x$. For each x^k we can choose $x_1^k, x_2^k, \dots, x_{n+1}^k \in S$ and $\alpha^k = (\alpha_1^k, \alpha_2^k, \dots, \alpha_{n+1}^k) \in \Delta_{n+1}$ such that

$$x^k = \sum_{i=1}^{n+1} \alpha_i^k x_i^k.$$

Since Δ_{n+1} is compact and S is compact we can extract subsequences of the sequences $\{\alpha^k\}$ and $\{x_i^k\}$, $i = 1,2, \dots, n+1$ which converge. Let $x_i^{k_l} \to x_i, i = 1,2, \dots, n+1$ and $\alpha^{k_l} \to \alpha$. As S and Δ_{n+1} are closed it follows that $x_i \in S$, $= 1,2, \dots, n+1$ and $\alpha \in \Delta_{n+1}$. Hence

$$x = \sum_{i=1}^{n+1} \alpha_i x_i \in \mathrm{co}S.$$

$\mathbf{cl}(\mathbf{co}S) = \mathbf{co}(\mathbf{cl}S)$?

Theorem If S is compact then $\mathrm{cl}(\mathrm{co}S) = \mathrm{co}(\mathrm{cl}S)$.

Proof As S is compact so $\mathrm{co}S$ is compact. Hence

$$\mathrm{cl}(\mathrm{co}S) = \mathrm{co}S.$$

As S is closed we have $S = \mathrm{cl}S$. Hence,

$$\mathrm{cl}(\mathrm{co}S) = \mathrm{co}S = \mathrm{co}(\mathrm{cl}S).$$

Theorem If S is bounded then $\mathrm{cl}(\mathrm{co}S) = \mathrm{co}(\mathrm{cl}S)$.

Proof As $S \subseteq \mathrm{cl}S$ we have $\mathrm{co}S \subseteq \mathrm{co}(\mathrm{cl}S)$. This implies

$$\mathrm{cl}(\mathrm{co}S) \subseteq \mathrm{cl}(\mathrm{co}(\mathrm{cl}S)). \qquad (1)$$

As S is bounded so $\mathrm{cl}S$ is compact. Hence by the previous theorem

$$\mathrm{cl}\big(\mathrm{co}(\mathrm{cl}S)\big) = \mathrm{co}(\mathrm{cl}(\mathrm{cl}S)) = \mathrm{co}(\mathrm{cl}S). \qquad (2)$$

Also we have

$$\mathrm{co}(\mathrm{cl}S) \subseteq \mathrm{cl}(\mathrm{co}S) \qquad (3)$$

From (1)-(3) we have

$$\mathrm{cl}(\mathrm{co}S) \subseteq \mathrm{co}(\mathrm{cl}S) \subseteq \mathrm{cl}(\mathrm{co}S).$$

Convex cone

A set $K \subseteq \mathbb{R}^n$ is said to be a cone if for $x \in K$ and $\alpha > 0$, we have

$$\alpha x \in K.$$

A convex cone K is a cone which is convex.

Theorem A cone K is convex if and only if for every $x, x' \in K$ we have $x + x' \in K$.

Proof Let *K* be a convex cone. Let $x, x' \in K$. As K is convex we have

$$\frac{1}{2}x + \frac{1}{2}x' \in K.$$

As *K* is a cone we have

$$x + x' = 2\left(\frac{1}{2}x + \frac{1}{2}x'\right) \in K.$$

Conversely, let $x, x' \in K$ and $\lambda \in [0,1]$. As *K* is a cone we have

$$\lambda x \in K, \text{and } (1-\lambda)x' \in K.$$

Hence by the assumption we have

$$\lambda x + (1-\lambda)x' \in K.$$

Conical Combinations and Conical Hull

Let $\{x_i\}_{i=1}^{k}$ be a finite set of points in $\mathbb{R}^n$. A conical combination of these points is a point of the form

$$x = \sum_{i=1}^{k} \lambda_i x_i,\ \lambda_i \geq 0, i = 1,2, \dots k.$$

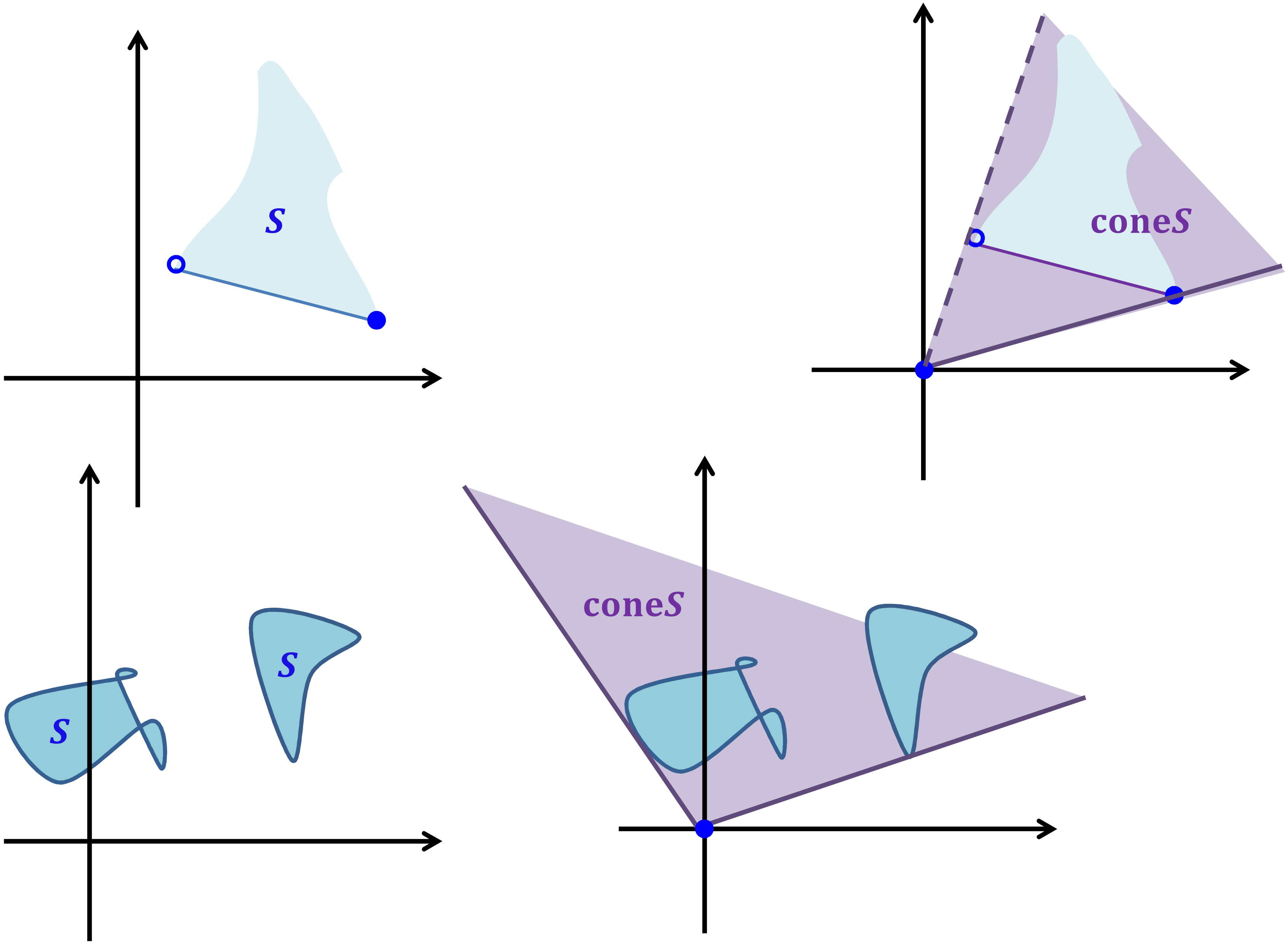
Conical hull of a set $S \subseteq \mathbb{R}^n$, denoted by cone$S$, is the set of all conical combinations of finite set of points of S.

Clearly, $0 \in$ cone$S$.

Theorem Conical hull of a set S in $\mathbb{R}^n$ is the intersection of all convex cone containing S and the origin.

Theorem Conical hull of a set S in $\mathbb{R}^n$ is the smallest convex cone containing S and the origin.

Conical Convex Hull of a Set



Conical hull in terms of convex hull

Theorem If S is a set in $\mathbb{R}^n$ then cone$S = \mathbb{R}^+(\mathrm{co}S)$.

Proof Let $x \in \mathrm{cone}S$. Then there exist $k \in \mathbb{N}$, $x_i \in S, \lambda_i \geq 0, i = 1,2,\dots,k$, such that

$$x = \sum_{i=1}^{k} \lambda_i x_i.$$

Let $\hat{\lambda} = \sum_{i=1}^{k} \lambda_i$. If $\hat{\lambda} = 0$, then $x = 0 \in \mathbb{R}^+(\mathrm{co}S)$. Let $\hat{\lambda} > 0$. Then

$$x = \hat{\lambda} \sum_{i=1}^{k} \frac{\lambda_i}{\hat{\lambda}} x_i \in \mathbb{R}^+(\mathrm{co}S).$$

Conversely, let $x \in \mathbb{R}^+(\mathrm{co}S)$. Then there exist $\alpha \geq 0$, $x_i \in S, \lambda_i \geq 0, i = 1,\dots,n+1, \sum_{i=1}^{k} \lambda_i = 1$, such that

$$x = \alpha \sum_{i=1}^{k} \lambda_i x_i = \sum_{i=1}^{k} \alpha\lambda_i x_i \in \text{cone}S.$$

Remark $0 \in \mathbb{R}^+(\mathrm{co}S) = \text{cone}S$.

Conical hull in terms of convex hull

Theorem If S is a set in $\mathbb{R}^n$ then cone$S = \mathbb{R}^+(\text{co}S) = \text{co}(\mathbb{R}^+ S)$.

Proof Let $x \in \mathbb{R}^+(\text{co}S)$. Then there exist $\alpha \geq 0$, $x_i \in S, \lambda_i \geq 0, i = 1, \dots, n+1, \sum_{i=1}^{k} \lambda_i = 1$, such that

$$x = \alpha \sum_{i=1}^{k} \lambda_i x_i = \sum_{i=1}^{k} \lambda_i (\alpha x_i) \in \text{co}(\mathbb{R}^+ S).$$

Conversely, let $x \in \text{co}(\mathbb{R}^+ S)$. Then there exist $u_i \in \mathbb{R}^+ S, \lambda_i \geq 0, i = 1, \dots, n+1, \sum_{i=1}^{k} \lambda_i = 1$, such that

$$x = \sum_{i=1}^{k} \lambda_i u_i \,.$$

As $u_i \in \mathbb{R}^+ S$, there exist $x_i \in S, \alpha_i \geq 0, i = 1, \dots, n+1$, such that $u_i = \alpha_i x_i$. Hence

$$x = \sum_{i=1}^{k} \lambda_i \alpha_i x_i \,.$$

If $\sum_{i=1}^{k} \lambda_i \alpha_i = 0$ then $x = 0 \in \mathbb{R}^+(\text{co}S)$. Let $\hat{\delta} = \sum_{i=1}^{k} \lambda_i \alpha_i > 0$. Then

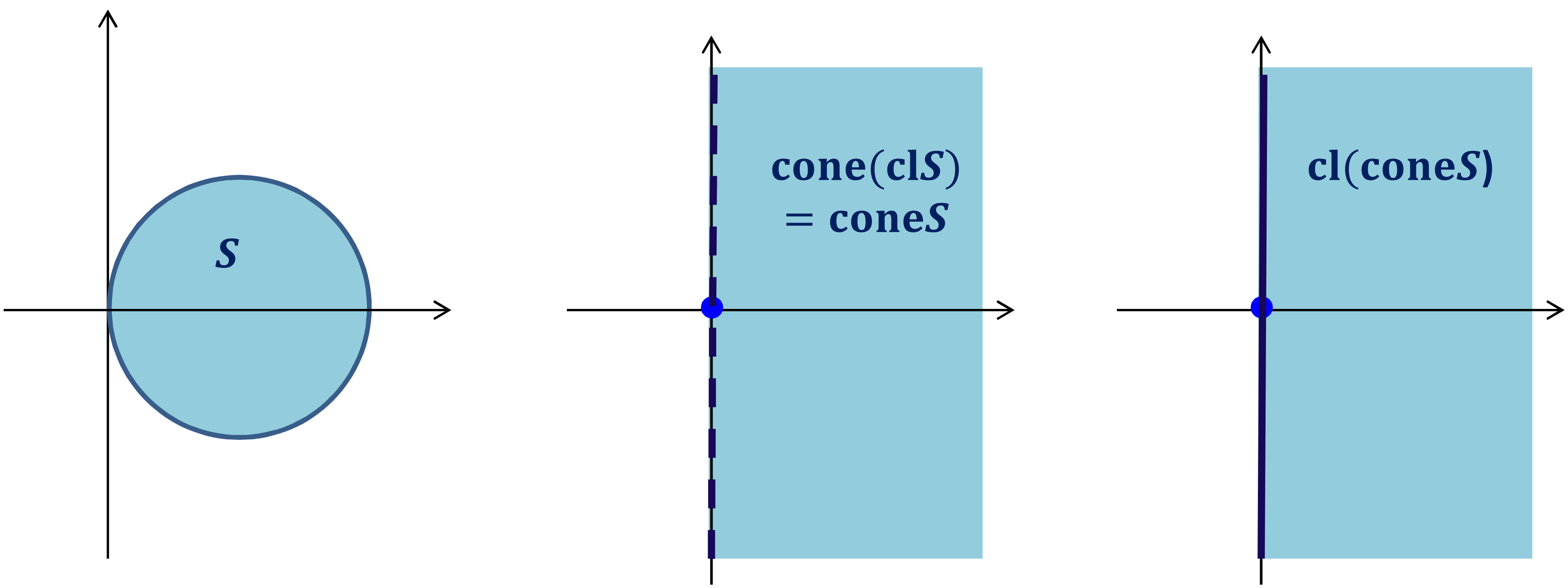
$$x = \hat{\delta} \sum_{i=1}^{k} \frac{\lambda_i \alpha_i}{\hat{\delta}} x_i \in \mathbb{R}^+(\text{co}S).$$

cl(coneS) and cone(clS)

We have $\mathrm{cl}(\mathrm{co}S) = \mathrm{co}(\mathrm{cl}S)$ if S is compact. But such a relation fails to hold even when S is compact when "co" is replaced by "cone".

Example $\mathrm{cl}(\mathrm{cone}S) \neq \mathrm{cone}(\mathrm{cl}S)$

Let $S = \{(x, y): (x-1)^2 + y^2 \leq 1\}$.



Closed conical hull

Closed conical hull of a nonempty set S in $\mathbb{R}^n$, denoted by $\overline{\text{cone}}S$, is defined as

$$\overline{\text{cone}}S := \text{cl}(\text{cone}S).$$

Theorem If S is a nonempty compact set in $\mathbb{R}^n$ such that $0 \notin \text{co}S$ then

$$\overline{\text{cone}}\, S = \text{cone}S.$$

Proof We need to show $\text{cl}(\text{cone}S) = \text{cone}S$.

It is enough to show $\text{cone}S$ is closed. We know $\text{cone}S = \mathbb{R}^+(\text{co}S)$.

So it is enough to show $\mathbb{R}^+(\text{co}S)$ is closed. Let $t_k x_k \in \mathbb{R}^+(\text{co}S)$ such that $t_k x_k \to y$. As $x_k \in \text{co}S$ and $0 \notin \text{co}S$ so $x_k \neq 0$. As S is compact so $\text{co}S$ is compact. Thus there exists a subsequence $x_{k_l} \to x \in \text{co}S$, $x \neq 0$. As $t_k x_k \to y$ we have

$$t_{k_l}\|x_{k_l}\| \to \|y\|.$$

As $\|x_{k_l}\| \to \|x\|$, $\|x_{k_l}\| \neq 0$, $\|x\| \neq 0$, we have

$$t_{k_l} = t_{k_l}\frac{\|x_{k_l}\|}{\|x_{k_l}\|} \to \frac{\|y\|}{\|x\|}.$$

Let $t = \frac{\|y\|}{\|x\|}$. As $x_{k_l} \to x$ and $t_{k_l} \to t$ we have $t_{k_l}x_{k_l} \to tx$. By uniqueness of limit we have $y = tx \in \mathbb{R}^+(\text{co}S)$.

Corollary If S is a nonempty compact set in $\mathbb{R}^n$ such that $0 \notin \text{co}S$ then

$$\text{cl}(\text{cone}S) = \text{cone}(\text{cl}S).$$