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M2112

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Book: Ward Cheney

Section: 3.1 (The Frechet Derivative)

Example: 4 (Page-117)

Let $X = Y = C[0,1]$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Define $f : X \to Y$ by the equation $f(x) = \phi \circ x$, where x is any element of $C[0,1]$. What is $f'(x)$?

Solution: Given that $f : X \to Y$ defined as $f(x) = \phi \circ x$ where $X = Y = C[0,1]$ for any $x \in C[0,1]$.

We need to find the Frechet derivative of f at x.

Consider,

$$
\begin{aligned}
(f(x+h) - f(x))(t) &= (\phi(x+h) - \phi(x))(t) \\
&= \phi(x+h)(t) - \phi(x)(t) \\
&= \phi(x(t) + h(t)) - \phi(x(t)) \\
&= \phi'(x(t) + \theta(t)h(t))\, h(t) \quad \text{--- (1)}
\end{aligned}
$$

[by classical Mean Value Theorem $\exists\ \theta \in (0,1)$ such that

$\phi(x(t) + h(t)) - \phi(x(t)) = \phi'(x(t) + \theta(t)h(t))\, h(t)$]

(Claim: A is frechet derivative of f at x)

From this (1), we define $A : X \to Y$ where $X = Y = C[0,1]$ as $Ah = (\phi' \circ x)(h)$ (pointwise multiplication) is a bounded linear operator.

Claim: A is bounded

$$
\begin{aligned}
\|Ah(t)\| &= \|(\phi' \circ x) \cdot h(t)\| \\
&= \|\phi'(x(t))\| \cdot \|h(t)\| \\
&\leq \|\phi' \circ x\| \cdot \|h\| \cdot \|t\| \\
\Rightarrow \|Ah\| &\leq \|\phi' \circ x\| \cdot \|h\| \\
&= M \cdot \|h\|
\end{aligned}
$$

Hence A is bounded

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Claim: A is linear i.e; To prove: $A(a_1x_1+a_2x_2) = a_1Ax_1 + a_2Ax_2$

Consider,

$$A(a_1x_1+a_2x_2)(t) = (\phi'\circ x)(a_1x_1+a_2x_2)(t)$$
$$= \phi'(x(t))\cdot(a_1x_1+a_2x_2)(t)$$
$$= \phi'(x(t))\cdot(a_1x_1(t)+a_2x_2(t))$$
$$= a_1\phi'(x(t))\cdot x_1(t) + a_2\phi'(x(t))\cdot x_2(t)$$
$$= a_1Ax_1 + a_2Ax_2$$

Hence, A is linear operator.

Claim: $\lim_{h\to 0} \dfrac{\|f(x+h)-f(x)-Ah\|}{\|h\|} = 0$

Consider,

$$(f(x+h)-f(x)-Ah)(t) = \phi'(x(t)+\theta(t)h(t))h(t) - \phi'(x(t))h(t)$$

(using ①)

$$\|f(x+h)-f(x)-Ah\|_{C[0,1]} = \sup_{t\in[0,1]} |\phi'(x(t)+\theta(t)h(t))h(t) - \phi'(x(t))h(t)|$$

$$\le \|\phi'\circ(x+\theta h) - \phi'\circ x\|\,\|h\|$$

then $\lim_{h\to 0}\dfrac{\|f(x+h)-f(x)-Ah\|}{\|h\|} \le \lim_{h\to 0}\|\phi'\circ(x+\theta h)-\phi'\circ x\|$

$= 0$ [$\because \phi'$ is differentiable $\Rightarrow \phi'$ is continuous]

Hence $Ah = (\phi'\circ x)(h)$ is frechet derivative of f.

—x—x—x—x—