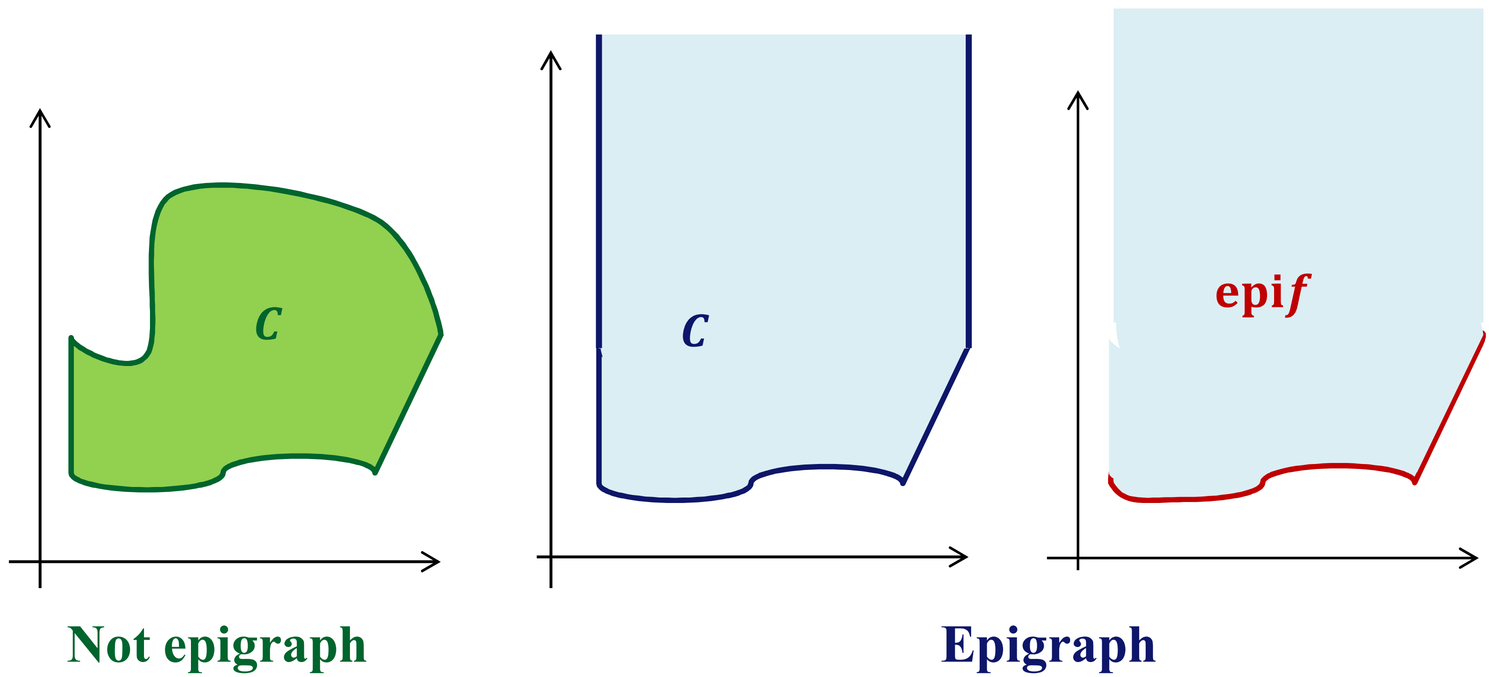
R-07
Convex and Nonsmooth Analysis

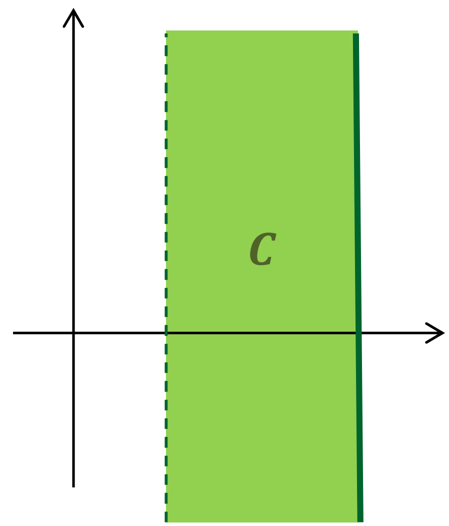
Are these sets epigraph of a function in Conv$\mathbb{R}^n$?



C must be unbounded from above, that is,

$$(x, r) \in C \quad \Rightarrow \quad (x, r') \in C, \forall\, r' > r.$$

Is this set the epigraph of a function in Conv$\mathbb{R}^n$?

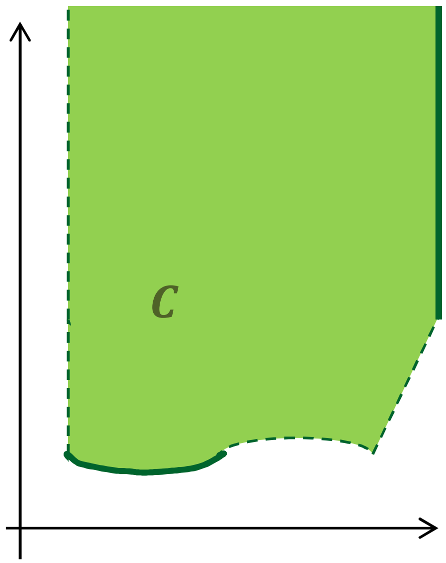


Epigraph of a function which is not in Conv$\mathbb{R}^n$

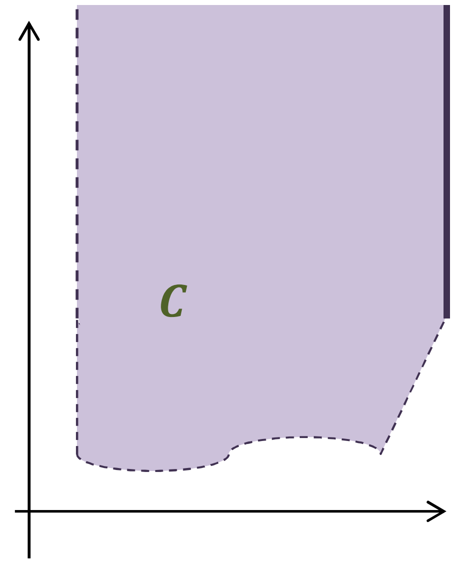
C must not contain any vertical downward half-line, that is,

$\{r \in \mathbb{R} : (x, r) \in C\}$ is minorized for all $x \in \mathbb{R}^n$.

Is this set the epigraph of a function in Conv$\mathbb{R}^n$?



Not epigraph



Not epigraph (But it is strict epigraph)

C must have a closed bottom, that is,

$$(x, r') \in C,\ r' \downarrow r \qquad \Rightarrow (x, r) \in C.$$

When is convex set C the epigraph (strict epigraph) of a function in Conv$\mathbb{R}^n$?

$$\{r \in \mathbb{R} : (x, r) \in C\} \text{ is minorized for all } x \in \mathbb{R}^n; \qquad (1)$$
$$(x, r) \in C \quad \Rightarrow \quad (x, r') \in C, \forall\, r' > r; \qquad (2)$$
$$(x, r') \in C,\ r' \downarrow r \quad \Rightarrow (x, r) \in C. \qquad (3)$$

A set C satisfying (1)-(3) is epigraph of a function.

Let C has an open bottom, that is,

$$(x, r) \in C \quad \Rightarrow \quad (x, r - \varepsilon) \in C, \text{ for some } \varepsilon > 0. \qquad (4)$$

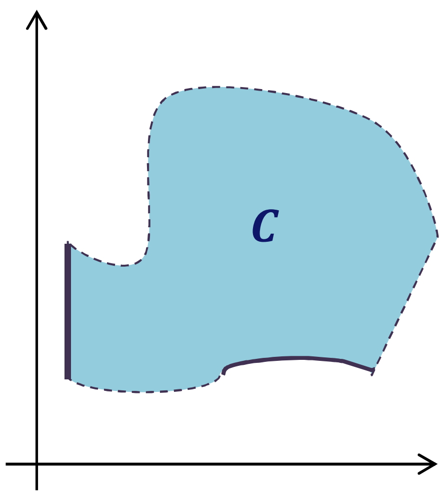
A set C satisfying (1),(2) and (4) is strict epigraph of a function.

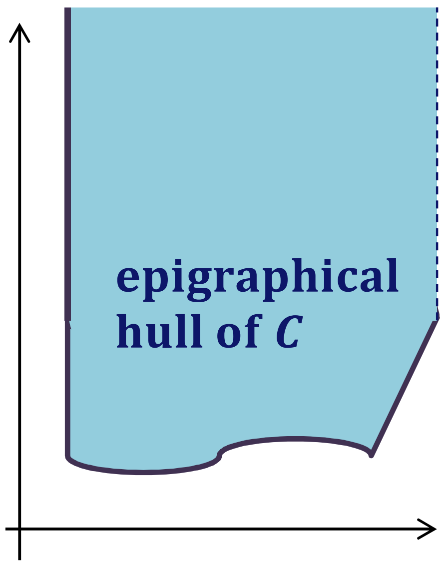
Epigraph is a union of closed upward half-lines.
Strict epigraph is a union of open upward half-lines.

Epigraphical hull of a set

Epigraphical hull of a set $C \subseteq \mathbb{R}^n \times \mathbb{R}$ is the smallest epigraph containing C.

This is achieved by stuffing in everything above C and closing the bottom.



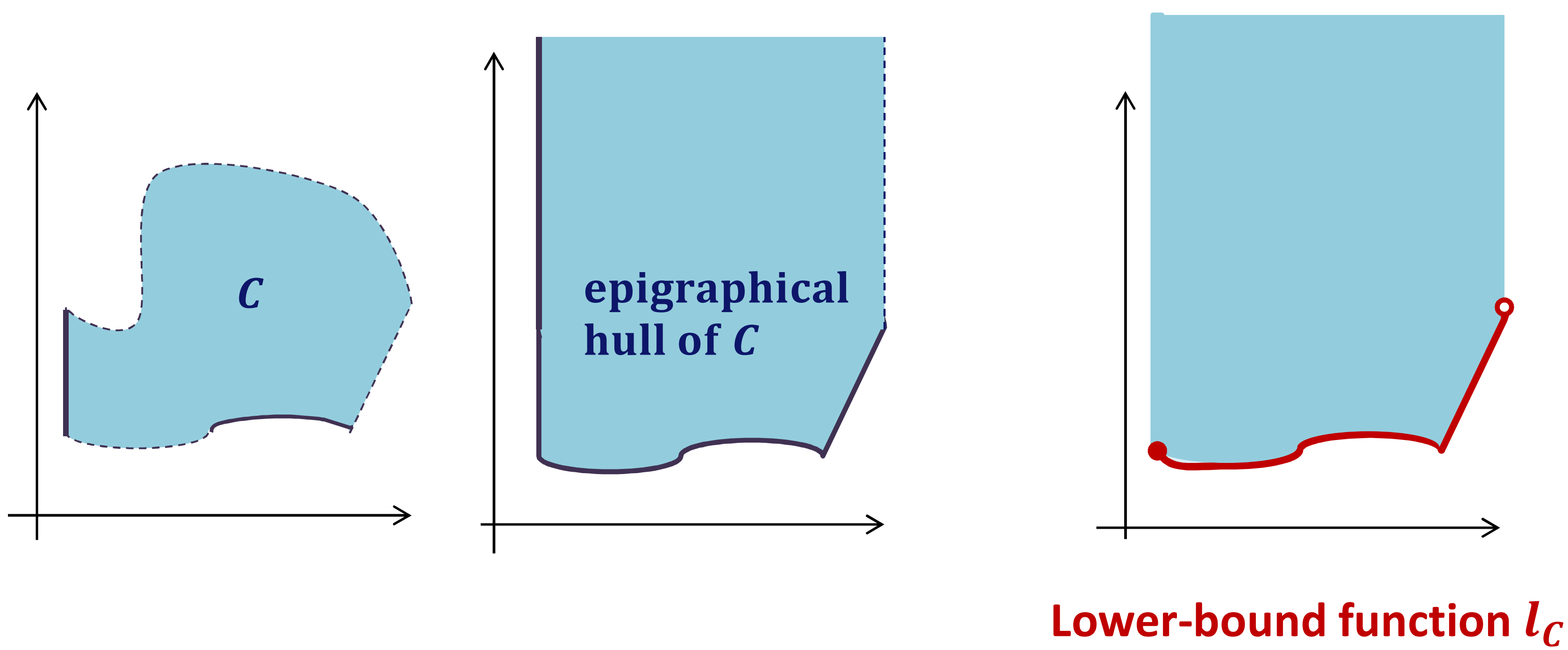


Lower Bound Function

The lower-bound function of a set $C \subseteq \mathbb{R}^n \times \mathbb{R}$ is the function $l_C : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined as

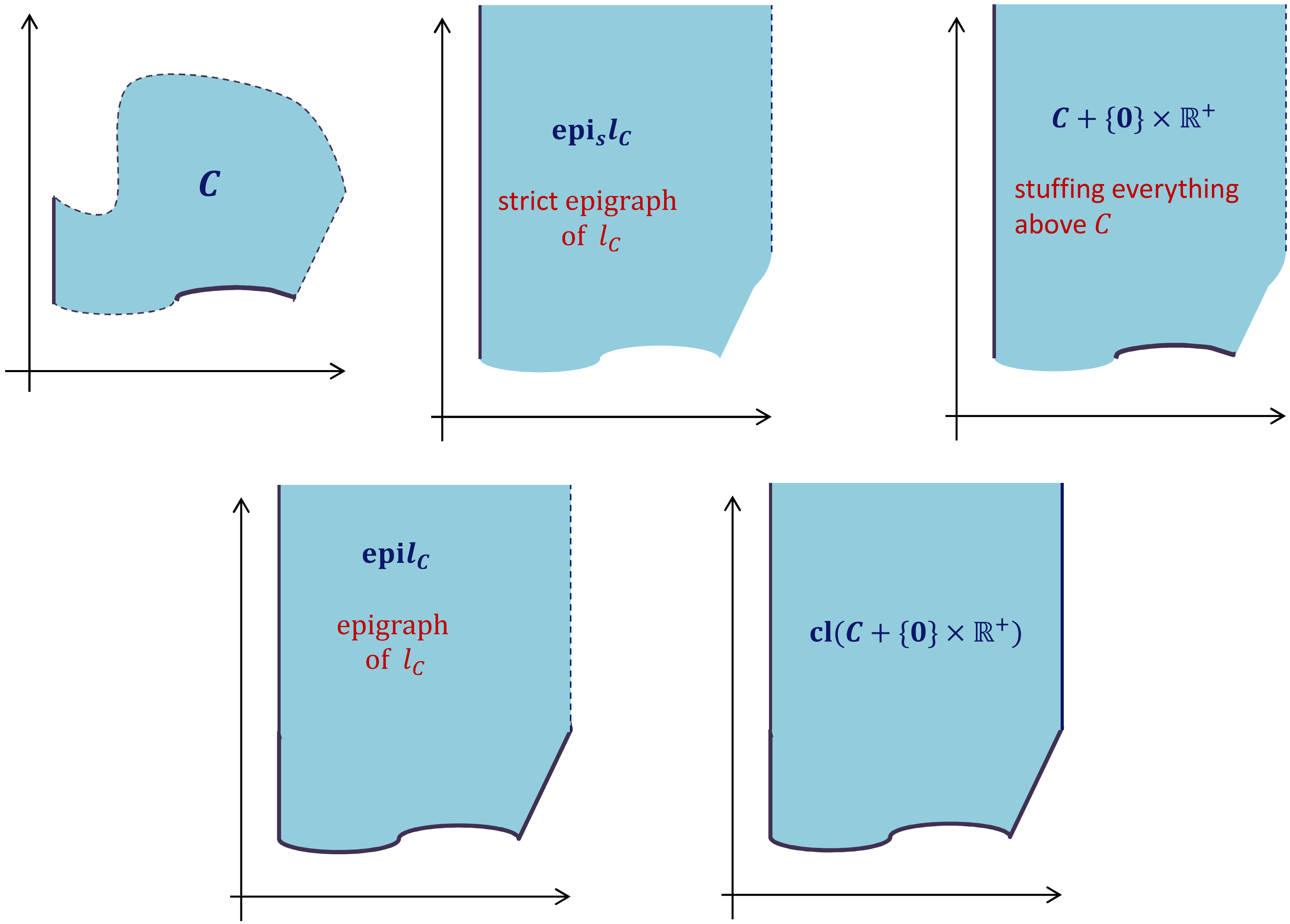
$$l_C(x) := \inf\{r \in \mathbb{R} : (x, r) \in C\}.$$

Clearly, epigraph of l_C is the epigraphical hull of C.



$$l_C(x) > -\infty \quad \Leftrightarrow \quad \{r \in \mathbb{R} : (x, r) \in C\} \text{ is minorized for all } x \in \mathbb{R}^n.$$

Lower Bound Function



$$\mathbf{epi}_s l_C \subset C + \{0\} \times \mathbb{R}^+ \subset \mathbf{epi}\, l_C \subset \mathbf{cl}(C + \{0\} \times \mathbb{R}^+)$$

Local Lipschitzness of a function at a point

Lemma Let $f \in \text{Conv}\mathbb{R}^n$ and suppose there are x_0, δ, m and M such that

$$m \leq f(x) \leq M, \text{ for all } x \in B(x_0, 2\delta).$$

Then f is Lipschitz on $B(x_0, \delta)$, that is, for all y and y' in $B(x_0, \delta)$,

$$|f(y) - f(y')| \leq \frac{M-m}{\delta} \|y - y'\|.$$

Proof For $y, y' \in B(x_0, \delta)$, $y \neq y'$ define

$$y'' = y' + \delta \frac{y'-y}{\|y'-y\|}. \quad (1)$$

Clearly, $\|y'' - x_0\| = \left\| y' + \delta \frac{y'-y}{\|y'-y\|} - x_0 \right\| \leq \|y' - x_0\| + \delta \leq 2\delta.$

Hence $y'' \in B(x_0, 2\delta)$. Also, from (1) we get

$$y' = \frac{\|y' - y\|}{\delta + \|y' - y\|} y'' + \frac{\delta}{\delta + \|y' - y\|} y.$$

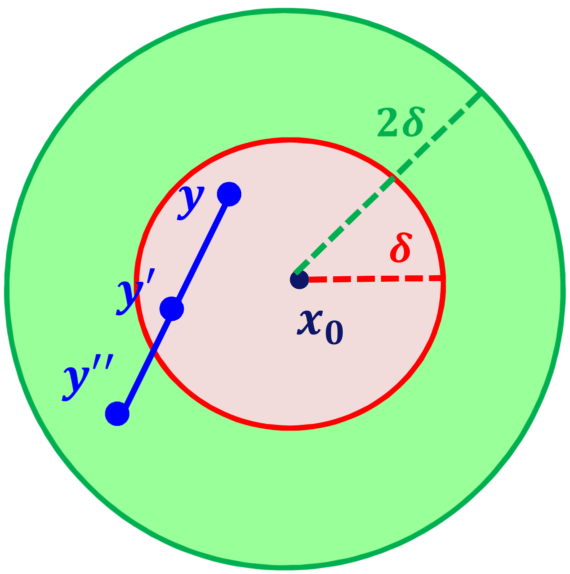
As f is convex we have

$$f(y') \leq \frac{\|y' - y\|}{\delta + \|y' - y\|} f(y'') + \frac{\delta}{\delta + \|y' - y\|} f(y).$$

Thus

$$f(y') - f(y) \leq \frac{\|y'-y\|}{\delta+\|y'-y\|} [f(y'') - f(y)]$$
$$\leq \frac{M-m}{\delta} \|y - y'\|.$$

Result follows by interchanging y and y'.



Lipschitzness of convex functions in every convex compact subset in the relative interior of domain

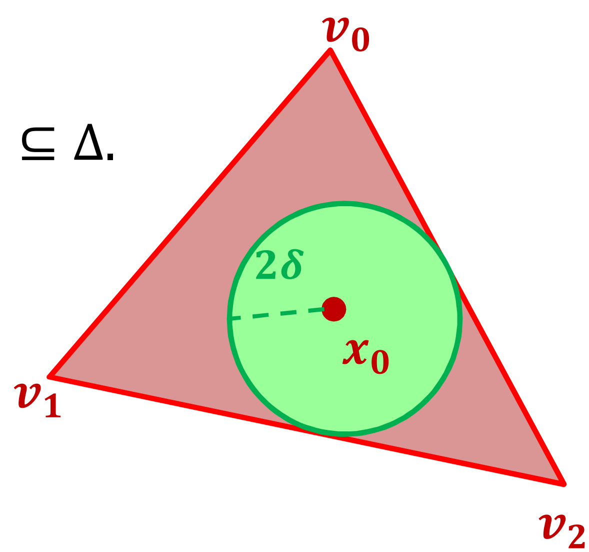
Theorem Let $f \in \text{Conv}\mathbb{R}^n$, let S be a convex compact subset of $\text{ri}\,\text{dom}\, f$. Then there exists $L = L(S) \geq 0$ such that

$$|f(x) - f(x')| \leq L\|x - x'\| \quad \text{for all } x, x' \in S.$$

Proof Without loss of generality we may assume $\text{ri}\,\text{dom}\, f = \text{int}\,\text{dom}\, f$.

Let $x_0 \in S$. Let $v_0, v_1, \dots, v_n \in \mathbb{R}^n$ be such that the simplex

$$\Delta = \text{co}\{v_0, v_1, \dots, v_n\} \subseteq \text{dom} f,$$
$$x_0 \in \text{int}\,\Delta.$$



Since $x_0 \in \text{int}\,\Delta$, we can choose $\delta > 0$ such that $B(x_0, 2\delta) \subseteq \Delta$.

Let $y \in B(x_0, 2\delta)$. Then there exists

$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n), \alpha_i \geq 0, i = 0,1, \dots n, \sum_{i=0}^n \alpha_i = 1$

such that

$$y = \sum_{i=0}^n \alpha_i v_i.$$

As f is convex we have

$$f(y) \leq \sum_{i=0}^n \alpha_i f(v_i).$$

As $f \in \text{Conv}\mathbb{R}^n$ it is bounded from below. Hence there exists $m_y \in \mathbb{R}$ such that $m_y \leq f(y)$. As this holds for every $y \in B(x_0, 2\delta)$ and $B(x_0, 2\delta)$ is bounded there exists $m \in \mathbb{R}$ such that

$$m \leq f(y), \forall y \in B(x_0, 2\delta).$$

Choose $M = \max\{f(v_0), f(v_1), \dots, f(v_n)\}$. Then we have

$$m \leq f(y) \leq M, \forall y \in B(x_0, 2\delta).$$

continued

Hence by the lemma there exists f is Lipschitz on $B(x_0, \delta)$, that is, there exists $L = L(x_0, \delta)$ such that for all y and y' in $B(x_0, \delta)$,

$$|f(y) - f(y')| \le L\|y - y'\|.$$

Clearly, $B(x_0, \delta) \subseteq \text{int dom } f$.

Since this condition holds for all $x_0 \in S$, the balls $B(x_0, \delta)$ for all $x_0 \in S$ is an open covering of S. As S is compact this cover has a finite subcover.

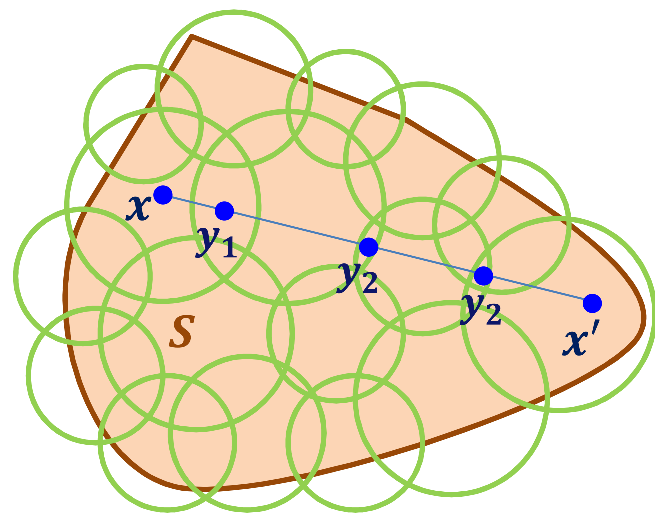
Let $(x_1, \delta_1, L_1), (x_2, \delta_2, L_2), \ldots, (x_k, \delta_k, L_k)$ correspond to the finite subcover.

Let $x, x' \in S$. Divide the line segment $[x, x']$ into many finite subsegments of end points

$$x = y_0, y_1, \ldots, y_i, \ldots, y_{l-1}, y_l = x'$$

such that $[y_i, y_{i+1}]$ is in the same ball for each $i = 0,1, \ldots l-1$. Suppose $[y_i, y_{i+1}] \subseteq B(x_r, \delta_r)$ for some $r \in \{1,2, \ldots, k\}$. Hence by the first part

$$|f(y_i) - f(y_{i+1})| \le L_r\|y_i - y_{i+1}\|.$$



Let $L = \max\{L_1, L_2, \ldots, L_k\}$. Then

$$|f(y_i) - f(y_{i+1})| \le L\|y_i - y_{i+1}\|.$$

Clearly, $\|x - x'\| = \sum_{i=0}^{l-1}\|y_i - y_{i+1}\|$ as all the points $y_0, y_1, \ldots, y_i, \ldots, y_{l-1}, y_l$ are collinear. Hence

$$|f(x) - f(x')| \le \sum_{i=0}^{l-1}|f(y_i) - f(y_{i+1})|$$
$$\le L\sum_{i=0}^{l-1}\|y_i - y_{i+1}\| = \|x - x'\|$$