VI.3:CALCULUS OF VARIATIONS AND LINEAR PROGRAMMING

Total marks:150 (Theory: 75, Internal Assessment: 25+ Practical: 50)
5 Periods (4 lectures +1 students' presentation),
Practicals(4 periods per week per student)

(1st&2nd Week)
Functionals, Some simple variational problems, The variation of a functional, A necessary condition for an extremum, The simplest variational problem, Euler's equation, A simple variable end point problem.
[1]: Chapter 1 (Sections 1, 3, 4 and 6).

(3rd&4th Week)
Introduction to linear programming problem, Graphical method of solution, Basic feasible solutions, Linear programming and Convexity.
[2]: Chapter 2 (Section 2.2), Chapter 3 (Sections 3.1, 3.2 and 3.9).

(5th& 6th Week)
Introduction to the simplex method, Theory of the simplex method, Optimality and Unboundedness.
[2]: Chapter 3 (Sections 3.3 and 3.4).

(7th& 8th Week)
The simplex tableau and examples, Artificial variables.
[2]: Chapter 3 (Sections 3.5 and 3.6).

(9th&10th Week)
Introduction to duality, Formulation of the dual problem, Primal-dual relationship, The duality theorem, The complementary slackness theorem.
[2]: Chapter 4 (Sections 4.1, 4.2, 4.4 and 4.5).

(11th&12th Week)
Transportation problem and its mathematical formulation, Northwest-corner method, Least-cost method and Vogel approximation method for determination of starting basic solution, Algorithm for solving transportation problem, Assignment problem and its mathematical formulation, Hungarian method for solving assignment problem.
[3]: Chapter 5 (Sections 5.1, 5.3 and 5.4)

PRACTICAL/LAB WORK TO BE PERFORMED ON A COMPUTER: (MODELLING OF THE FOLLOWING PROBLEMS USING EXCEL SOLVER/LINGO/MATHEMATICA, ETC.)

(i) Formulating and solving linear programming models on a spreadsheet using excel solver.
[2]: Appendix E and Chapter 3 (Examples 3.10.1 and 3.10.2).
[4]: Chapter 3 (Section 3.5 with Exercises 3.5-2 to 3.5-5)

(ii) Finding solution by solving its dual using excel solver and giving an interpretation of the dual.
[2]: Chapter 4 (Examples 4.3.1 and 4.4.2)

(iii) Using the excel solver table to find allowable range for each objective function coefficient, and the allowable range for each right-hand side.
[4]: Chapter 6 (Exercises 6.8-1 to 6.8-5).

(iv) Formulating and solving transportation and assignment models on a spreadsheet using solver.
[4]: Chapter 8 (**CASE 8.1**: Shipping Wood to Market, **CASE 8.3**: Project Pickings).

From the Metric space paper, exercises similar to those given below:

1. Calculate d(x,y) for the following metrics

   (i) X=**R**, d(x,y)=lx-yl, (ii) X=$\mathbf{R}^3$, d(x,y)= $(\sum(x_i-y_i)^2)^{1/2}$

   x: 0, 1, π, e x: (0,1,-1), (1,2,π), (2,-3,5)
   y: 1, 2, ½, √2 y: (1, 2, .5), (e,2,4), (-2,-3,5)

   (iii) X=C[0,1], d(f,g)= sup lf(x)-g(x)l

   f(x): x^2, sin x, tan x
   g(x): x , lxl, cos x

2. Draw open balls of the above metrics with centre and radius of your choice.
3. Find the fixed points for the following functions

   f(x)=x^2 , g(x)= sin x, h(x)= cos x in X=[-1, 1],

   f(x,y)= (sin x, cos y), g(x,y) = (x^2 , y^2) in X= { (x,y): $x^2+y^2 \leq 1$}, under the Euclidean metrics on **R** and $\mathbf{R}^2$ respectively.

4. Determine the compactness and connectedness by drawing sets in R^2.

REFERENCES:

[1]. I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Dover Publications, Inc., New York, 2000.
[2]. Paul R. Thie and Gerard E. Keough, *An Introduction to Linear Programming and Game Theory*, Third Edition, John Wiley & Sons, Inc., Hoboken, New Jersey, 2008.
[3]. Hamdy A. Taha, *Operations Research*: *An Introduction*, Ninth Edition, Prentice Hall, 2011.
[4]. Frederick S. Hillier and Gerald J. Lieberman, *Introduction to Operations Research*, Ninth Edition, McGraw-Hill, Inc., New York, 2010.

SUGGESTED READING:

[1].R. Weinstock, *Calculus of Variations*, Dover Publications, Inc. New York, 1974.
[2].M. L. Krasnov, G. I. Makarenko and A. I. Kiselev, *Problems and Exercises in the Calculus of Variations*, Mir Publishers, Moscow, 1975.
[3].Mokhtar S. Bazaraa, John J. Jarvis and Hanif D, Sherali, *Linear Programming and Network Flows*, Fourth Edition, John Wiley & Sons, Inc., Hoboken, New Jersey, 2010.
[4].G. Hadley, *Linear Programming*, Narosa Publishing House, New Delhi, 2002.

Minutes of the meeting held on January 08, 2016 at 2.00 PM in room no. 314, Arts Faculty, South Campus, University of Delhi, to discuss the paper: **DC-I(xviii), Calculus of variations and linear programming**, attended by various teachers of the colleges, who are teaching this course to B.Sc.(H) Mathematics VI Semester.

Text books:

1. I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Dover Publications, Inc., New York, 2000.
2. Paul R. Thie and G. E. Keough, *An Introduction to Linear Programming and Game Theory*, Third Edition, John Wiley & Sons, Inc., Hoboken, 2008.
3. Hamdy A. Taha, *Operations Research*: *An Introduction*, Ninth Edition, Prentice Hall, 2011.

From [1]: Chapter 1:

Section 1: Examples of functionals from page 1 to first paragraph of page 3.
Section 3: Pages 9 –11, except Lemma 3. Page 13, Theorem 2 (statement only).
Section 4: Subsection 4.1 up to Theorem 1 (with proof), subsection 4.2 including all four cases of the Euler's equation and examples (pages 18-22).
Section 6: Example on brachistochrone problem, page 26.

From [2]: Chapters 2, 3 and 4:

Section 2.2: Examples 2.2.1, 2.2.2 and problem set 2.2 (pages 17-18), 1 to 5.
Section 3.1: Example 3.1.1 and problem set 3.1 (pages 60-61), 1, 2, 3[(a) to (e)].
Section 3.2: Examples 3.2.1, 3.2.2 and problem set 3.2 (pages 70-71), 1, 2, 3, 4[(a), (b)], also no geometric representations.
Section 3.3: Complete and problem set 3.3 (pages 76-77), 1 to 4.
Section 3.4: Complete with Theorem 3.4.3 (statement only), and problem set 3.4 (pages 83-84), 1, 2.
Section 3.5: Examples 3.5.1, 3.5.2 and problem set 3.5 (pages 89-90), 2[(b) to (f)], 6(a).
Section 3.6: Examples 3.6.1, 3.6.2 and problem set 3.6 (pages 98-100), 2[(a) to (d)].
Section 3.9 and Section 4.1: Complete, except problem sets.
Section 4.2: Complete with problem set 4.2 (pages 130-131), 1.
Section 4.4: Theorem 4.4.1, Corollary 4.4.1, 4.4.2, 4.4.3, Theorem 4.4.2(statement only), Corollary 4.4.4. Simple problems based on the duality theorem.
Section 4.5: Theorem 4.5.1 (statement only), Examples 4.5.1, 4.5.2 and problem set 4.5 (page 158), 1, 2.

From [3]: Chapter 5: Sections 5.1, 5.3 and 5.4 complete with emphasis on methods and problems.

CALCULUS OF VARIATIONS

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ELEMENTS OF THE THEORY

I. Functionals. Some Simple Variational Problems

Variable quantities called *functionals* play an important role in many problems arising in analysis, mechanics, geometry, etc. By a *functional*, we mean a correspondence which assigns a definite (real) number to each function (or curve) belonging to some class. Thus, one might say that a functional is a kind of function, where the independent variable is itself a function (or curve). The following are examples of functionals:

1. Consider the set of all rectifiable plane curves.[1] A definite number is associated with each such curve, namely, its length. Thus, the length of a curve is a functional defined on the set of rectifiable curves.
2. Suppose that each rectifiable plane curve is regarded as being made out of some homogeneous material. Then if we associate with each such curve the ordinate of its center of mass, we again obtain a functional.
3. Consider all possible paths joining two given points A and B in the plane. Suppose that a particle can move along any of these paths, and let the particle have a definite velocity $v(x, y)$ at the point (x, y). Then we obtain a functional by associating with each path the time the particle takes to traverse the path.

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[1] In analysis, the *length* of a curve is defined as the limiting length of a polygonal line inscribed in the curve (i.e., with vertices lying on the curve) as the maximum length of the chords forming the polygonal line goes to zero. If this limit exists and is finite, the curve is said to be *rectifiable.*

4. Let $y(x)$ be an arbitrary continuously differentiable function, defined on the interval $[a, b]$.[2] Then the formula
$$J[y] = \int_a^b y'^2(x)\,dx$$
defines a functional on the set of all such functions $y(x)$.

5. As a more general example, let $F(x, y, z)$ be a continuous function of three variables. Then the expression
$$J[y] = \int_a^b F[x, y(x), y'(x)]\,dx, \tag{1}$$
where $y(x)$ ranges over the set of all continuously differentiable functions defined on the interval $[a, b]$, defines a functional. By choosing different functions $F(x, y, z)$, we obtain different functionals. For example, if
$$F(x, y, z) = \sqrt{1 + z^2},$$
$J[y]$ is the length of the curve $y = y(x)$, as in the first example, while if
$$F(x, y, z) = z^2,$$
$J[y]$ reduces to the case considered in the fourth example. In what follows, we shall be concerned mainly with functionals of the form (1).

Particular instances of problems involving the concept of a functional were considered more than three hundred years ago, and in fact, the first important results in this area are due to Euler (1707–1783). Nevertheless, up to now, the "calculus of functionals" still does not have methods of a generality comparable to the methods of classical analysis (i.e., the ordinary "calculus of functions"). The most developed branch of the "calculus of functionals" is concerned with finding the maxima and minima of functionals, and is called the "calculus of variations." Actually, it would be more appropriate to call this subject the "calculus of variations in the narrow sense," since the significance of the concept of the variation of a functional is by no means confined to its applications to the problem of determining the extrema of functionals.

We now indicate some typical examples of *variational problems*, by which we mean problems involving the determination of maxima and minima of functionals.

1. *Find the shortest plane curve joining two points A and B, i.e., find the curve* $y = y(x)$ *for which the functional*
$$\int_a^b \sqrt{1 + y'^2}\,dx$$
achieves its minimum. The curve in question turns out to be the straight line segment joining A and B.

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[2] By $[a, b]$ is meant the closed interval $a \leqslant x \leqslant b$.

2. Let A and B be two fixed points. Then the time it takes a particle to slide under the influence of gravity along some path joining A and B depends on the choice of the path (curve), and hence is a functional. The curve such that the particle takes the least time to go from A to B is called the *brachistochrone.* The brachistochrone problem was posed by John Bernoulli in 1696, and played an important part in the development of the calculus of variations. The problem was solved by John Bernoulli, James Bernoulli, Newton, and L'Hospital. The brachistochrone turns out to be a cycloid, lying in the vertical plane and passing through A and B (cf. p. 26).

3. The following variational problem, called the *isoperimetric problem,* was solved by Euler: *Among all closed curves of a given length l, find the curve enclosing the greatest area.* The required curve turns out to be a circle.

All of the above problems involve functionals which can be written in the form

$$\int_a^b F(x, y, y')\,dx.$$

Such functionals have a "localization property" consisting of the fact that if we divide the curve $y = y(x)$ into parts and calculate the value of the functional for each part, the sum of the values of the functional for the separate parts equals the value of the functional for the whole curve. It is just these functionals which are usually considered in the calculus of variations. As an example of a "nonlocal functional," consider the expression

$$\frac{\int_a^b x\sqrt{1 + y'^2}\,dx}{\int_a^b \sqrt{1 + y'^2}\,dx},$$

which gives the abscissa of the center of mass of a curve $y = y(x)$, $a \leqslant x \leqslant b$, made out of some homogeneous material.

An important factor in the development of the calculus of variations was the investigation of a number of mechanical and physical problems, e.g., the brachistochrone problem mentioned above. In turn, the methods of the calculus of variations are widely applied in various physical problems. It should be emphasized that the application of the calculus of variations to physics does not consist merely in the solution of individual, albeit very important problems. The so-called "variational principles," to be discussed in Chapters 4 and 7, are essentially a manifestation of very general physical laws, which are valid in diverse branches of physics, ranging from classical mechanics to the theory of elementary particles.

To understand the basic meaning of the problems and methods of the calculus of variations, it is very important to see how they are related to in general, the functional will not be continuous if we use the norm introduced in the space $\mathscr{C}$,[5] even though it is continuous in the norm of the space $\mathscr{D}_1$. Since we want to be able to use ordinary analytic methods, e.g., passage to the limit, then, given a functional, it is reasonable to choose a function space such that the functional is continuous.

Remark 3. So far, we have talked about linear spaces and functionals defined on them. However, in many variational problems, we have to deal with functionals defined on sets of functions which do not form linear spaces. In fact, the set of functions (or curves) satisfying the constraints of a given variational problem, called the *admissible functions* (or *admissible curves*), is in general not a linear space. For example, the admissible curves for the "simplest" variational problem (see Sec. 4) are the smooth plane curves passing through two fixed points, and the sum of two such curves does not pass through the two points. Nevertheless, the concept of a normed linear space and the related concepts of the distance between functions, continuity of functionals, etc., play an important role in the calculus of variations. A similar situation is encountered in elementary analysis, where, in dealing with functions of n variables, it is convenient to use the concept of an n-dimensional Euclidean space $\mathscr{E}_n$, even though the domain of definition of a function may not be a linear subspace of $\mathscr{E}_n$.

3. The Variation of a Functional. A Necessary Condition for an Extremum

3.1. In this section, we introduce the concept of the *variation* (or *differential*) of a functional, analogous to the concept of the differential of a function of n variables. The concept will then be used to find extrema of functionals. First, we give some preliminary facts and definitions.

DEFINITION. *Given a normed linear space $\mathscr{R}$, let each element $h \in \mathscr{R}$ be assigned a number $\varphi[h]$, i.e., let $\varphi[h]$ be a functional defined on $\mathscr{R}$. Then $\varphi[h]$ is said to be a (continuous) linear functional if*

1. $\varphi[\alpha h] = \alpha\varphi[h]$ *for any $h \in \mathscr{R}$ and any real number α;*
2. $\varphi[h_1 + h_2] = \varphi[h_1] + \varphi[h_2]$ *for any $h_1, h_2 \in \mathscr{R}$;*
3. $\varphi[h]$ *is continuous (for all $h \in \mathscr{R}$).*

Example 1. If we associate with each function $h(x) \in \mathscr{C}(a, b)$ its value at a fixed point x_0 in $[a, b]$, i.e., if we define the functional $\varphi[h]$ by the formula

$$\varphi[h] = h(x_0),$$

then $\varphi[h]$ is a linear functional on $\mathscr{C}(a, b)$.

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[5] Arc length is a typical example of such a functional. For every curve, we can find another curve arbitrarily close to the first in the sense of the norm of the space $\mathscr{C}$, whose length differs from that of the first curve by a factor of 10, say.

Example 2. The integral

$$\varphi[h] = \int_a^b h(x)\, dx$$

defines a linear functional on $\mathscr{C}(a, b)$.

Example 3. The integral

$$\varphi[h] = \int_a^b \alpha(x)h(x)\, dx,$$

where $\alpha(x)$ is a fixed function in $\mathscr{C}(a, b)$, defines a linear functional on $\mathscr{C}(a, b)$.

Example 4. More generally, the integral

$$\varphi[h] = \int_a^b [\alpha_0(x)h(x) + \alpha_1(x)h'(x) + \cdots + \alpha_n(x)h^{(n)}(x)]\, dx, \tag{6}$$

where the $\alpha_i(x)$ are fixed functions in $\mathscr{C}(a, b)$, defines a linear functional on $\mathscr{D}_n(a, b)$.

Suppose the linear functional (6) vanishes for all $h(x)$ belonging to some class. Then what can be said about the functions $\alpha_i(x)$? Some typical results in this direction are given by the following lemmas:

LEMMA 1. *If* $\alpha(x)$ *is continuous in* $[a, b]$, *and if*

$$\int_a^b \alpha(x)h(x)\, dx = 0$$

for every function $h(x) \in \mathscr{C}(a, b)$ *such that* $h(a) = h(b) = 0$, *then* $\alpha(x) = 0$ *for all* x *in* $[a, b]$.

Proof. Suppose the function $\alpha(x)$ is nonzero, say positive, at some point in $[a, b]$. Then $\alpha(x)$ is also positive in some interval $[x_1, x_2]$ contained in $[a, b]$. If we set

$$h(x) = (x - x_1)(x_2 - x)$$

for x in $[x_1, x_2]$ and $h(x) = 0$ otherwise, then $h(x)$ obviously satisfies the conditions of the lemma. However,

$$\int_a^b \alpha(x)h(x)\, dx = \int_{x_1}^{x_2} \alpha(x)(x - x_1)(x_2 - x)\, dx > 0,$$

since the integrand is positive (except at x_1 and x_2). This contradiction proves the lemma.

Remark. The lemma still holds if we replace $\mathscr{C}(a, b)$ by $\mathscr{D}_n(a, b)$. To see this, we use the same proof with

$$h(x) = [(x - x_1)(x_2 - x)]^{n+1}$$

for x in $[x_1, x_2]$ and $h(x) = 0$ otherwise.

LEMMA 2. *If* $\alpha(x)$ *is continuous in* $[a, b]$, *and if*

$$\int_a^b \alpha(x)h'(x)\,dx = 0$$

for every function $h(x) \in \mathscr{D}_1(a, b)$ *such that* $h(a) = h(b) = 0$, *then* $\alpha(x) = c$ *for all* x *in* $[a, b]$, *where* c *is a constant.*

Proof. Let c be the constant defined by the condition

$$\int_a^b [\alpha(x) - c]\,dx = 0,$$

and let

$$h(x) = \int_a^x [\alpha(\xi) - c]\,d\xi,$$

so that $h(x)$ automatically belongs to $\mathscr{D}_1(a, b)$ and satisfies the conditions $h(a) = h(b) = 0$. Then on the one hand,

$$\int_a^b [\alpha(x) - c]h'(x)\,dx = \int_a^b \alpha(x)h'(x)\,dx - c[h(b) - h(a)] = 0,$$

while on the other hand,

$$\int_a^b [\alpha(x) - c]h'(x)\,dx = \int_a^b [\alpha(x) - c]^2\,dx.$$

It follows that $\alpha(x) - c = 0$, i.e., $\alpha(x) = c$, for all x in $[a, b]$.

The next lemma will be needed in Chapter 8:

LEMMA 3. *If* $\alpha(x)$ *is continuous in* $[a, b]$, *and if*

$$\int_a^b \alpha(x)h''(x)\,dx = 0$$

for every function $h(x) \in \mathscr{D}_2(a, b)$ *such that* $h(a) = h(b) = 0$ *and* $h'(a) = h'(b) = 0$, *then* $\alpha(x) = c_0 + c_1x$ *for all* x *in* $[a, b]$, *where* c_0 *and* c_1 *are constants.*

Proof. Let c_0 and c_1 be defined by the conditions

$$\begin{aligned} \int_a^b [\alpha(x) - c_0 - c_1x]\,dx &= 0, \\ \int_a^b dx \int_a^x [\alpha(\xi) - c_0 - c_1\xi]\,d\xi &= 0, \end{aligned} \tag{7}$$

and let

$$h(x) = \int_a^x d\xi \int_a^\xi [\alpha(t) - c_0 - c_1t]\,dt,$$

so that $h(x)$ automatically belongs to $\mathscr{D}_2(a, b)$ and satisfies the conditions $h(a) = h(b) = 0$, $h'(a) = h'(b) = 0$. Then on the one hand,

$$\begin{aligned} &\int_a^b [\alpha(x) - c_0 - c_1x]h''(x)\,dx \\ &\qquad = \int_a^b \alpha(x)h''(x)\,dx - c_0[h'(b) - h'(a)] - c_1 \int_a^b xh''(x)\,dx \\ &\qquad = -c_1[bh'(b) - ah'(a)] - c_1[h(b) - h(a)] = 0, \end{aligned}$$

while on the other hand,

$$\int_a^b [\alpha(x) - c_0 - c_1x]h''(x)\,dx = \int_a^b [\alpha(x) - c_0 - c_1x]^2\,dx = 0.$$

It follows that $\alpha(x) - c_0 - c_1x = 0$, i.e., $\alpha(x) = c_0 + c_1x$, for all x in $[a, b]$.

LEMMA 4. *If* $\alpha(x)$ *and* $\beta(x)$ *are continuous in* $[a, b]$, *and if*

$$\int_a^b [\alpha(x)h(x) + \beta(x)h'(x)]\,dx = 0 \tag{8}$$

for every function $h(x) \in \mathscr{D}_1(a, b)$ *such that* $h(a) = h(b) = 0$, *then* $\beta(x)$ *is differentiable, and* $\beta'(x) = \alpha(x)$ *for all* x *in* $[a, b]$.

Proof. Setting

$$A(x) = \int_a^x \alpha(\xi)\,d\xi,$$

and integrating by parts, we find that

$$\int_a^b \alpha(x)h(x)\,dx = -\int_a^b A(x)h'(x)\,dx,$$

i.e., (8) can be rewritten as

$$\int_a^b [-A(x) + \beta(x)]h'(x)\,dx = 0.$$

But, according to Lemma 2, this implies that

$$\beta(x) - A(x) = \text{const},$$

and hence by the definition of $A(x)$,

$$\beta'(x) = \alpha(x),$$

for all x in $[a, b]$, as asserted. We emphasize that the differentiability of the function $\beta(x)$ was not assumed in advance.

3.2. We now introduce the concept of the *variation* (or *differential*) of a functional. Let $J[y]$ be a functional defined on some normed linear space, and let

$$\Delta J[h] = J[y + h] - J[y]$$

be its *increment*, corresponding to the increment $h = h(x)$ of the "independent variable" $y = y(x)$. If y is fixed, $\Delta J[h]$ is a functional of h, in general a nonlinear functional. Suppose that

$$\Delta J[h] = \varphi[h] + \varepsilon\|h\|,$$

where $\varphi[h]$ is a linear functional and $\varepsilon \to 0$ as $\|h\| \to 0$. Then the functional $J[y]$ is said to be *differentiable*, and the *principal linear part* of the increment

$\Delta J[h]$, i.e., the linear functional $\varphi[h]$ which differs from $\Delta J[h]$ by an infinitesimal of order higher than 1 relative to $\|h\|$, is called the *variation* (or *differential*) of $J[y]$ and is denoted by $\delta J[h]$.[6]

THEOREM 1. *The differential of a differentiable functional is unique.*

Proof. First, we note that if $\varphi[h]$ is a linear functional and if

$$\frac{\varphi[h]}{\|h\|} \to 0$$

as $\|h\| \to 0$, then $\varphi[h] \equiv 0$, i.e., $\varphi[h] = 0$ for all h. In fact, suppose $\varphi[h_0] \neq 0$ for some $h_0 \neq 0$. Then, setting

$$h_n = \frac{h_0}{n}, \qquad \lambda = \frac{\varphi[h_0]}{\|h_0\|},$$

we see that $\|h_n\| \to 0$ as $n \to \infty$, but

$$\lim_{n\to\infty} \frac{\varphi[h_n]}{\|h_n\|} = \lim_{n\to\infty} \frac{n\varphi[h_0]}{n\|h_0\|} = \lambda \neq 0,$$

contrary to hypothesis.

Now, suppose the differential of the functional $J[y]$ is not uniquely defined, so that

$$\Delta J[h] = \varphi_1[h] + \varepsilon_1\|h\|,$$
$$\Delta J[h] = \varphi_2[h] + \varepsilon_2\|h\|,$$

where $\varphi_1[h]$ and $\varphi_2[h]$ are linear functionals, and $\varepsilon_1, \varepsilon_2 \to 0$ as $\|h\| \to 0$. This implies

$$\varphi_1[h] - \varphi_2[h] = \varepsilon_2\|h\|$$

and hence $\varphi_1[h] - \varphi_2[h]$ is an infinitesimal of order higher than 1 relative to $\|h\|$. But since $\varphi_1[h] - \varphi_2[h]$ is a linear functional, it follows from the first part of the proof that $\varphi_1[h] - \varphi_2[h]$ vanishes identically, as asserted.

Next, we use the concept of the variation (or) differential of a functional to establish a necessary condition for a functional to have an extremum. We begin by recalling the corresponding concepts from analysis. Let $F(x_1, \ldots, x_n)$ be a differentiable function of n variables. Then $F(x_1, \ldots, x_n)$ is said to have a (*relative*) *extremum* at the point $(\hat{x}_1, \ldots, \hat{x}_n)$ if

$$\Delta F = F(x_1, \ldots, x_n) - F(\hat{x}_1, \ldots, \hat{x}_n)$$

has the same sign for all points $(x_1, \ldots, x_n)$ belonging to some neighborhood of $(\hat{x}_1, \ldots, \hat{x}_n)$, where the extremum $F(\hat{x}_1, \ldots, \hat{x}_n)$ is a *minimum* if $\Delta F \geqslant 0$ and a *maximum* if $\Delta F \leqslant 0$.

Analogously, we say that the functional $J[y]$ has a (*relative*) *extremum* for $y = \hat{y}$ if $J[y] - J[\hat{y}]$ does not change its sign in some neighborhood of

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[6] Strictly speaking, of course, the increment and the variation of $J[y]$, are functionals of two arguments y and h, and to emphasize this fact, we might write $\Delta J[y; h] = \delta J[y; h] + \varepsilon\|h\|$.

the curve $y = \hat{y}(x)$. Subsequently, we shall be concerned with functionals defined on some set of continuously differentiable functions, and the functions themselves can be regarded either as elements of the space $\mathscr{C}$ or elements of the space $\mathscr{D}_1$. Corresponding to these two possibilities, we can define two kinds of extrema: We shall say that the functional $J[y]$ has a *weak extremum* for $y = \hat{y}$ if there exists an $\varepsilon > 0$ such that $J[y] - J[\hat{y}]$ has the same sign for all y in the domain of definition of the functional which satisfy the condition $\|y - \hat{y}\|_1 < \varepsilon$, where $\| \quad \|_1$ denotes the norm in the space $\mathscr{D}_1$. On the other hand, we shall say that the functional $J[y]$ has a *strong extremum* for $y = \hat{y}$ if there exists an $\varepsilon > 0$ such that $J[y] - J[\hat{y}]$ has the same sign for all y in the domain of definition of the functional which satisfy the condition $\|y - \hat{y}\|_0 < \varepsilon$, where $\| \quad \|_0$ denotes the norm in the space $\mathscr{C}$. It is clear that every strong extremum is simultaneously a weak extremum, since if $\|y - \hat{y}\|_1 < \varepsilon$, then $\|y - \hat{y}\|_0 < \varepsilon$, *a fortiori*, and hence, if $J[\hat{y}]$ is an extremum with respect to all y such that $\|y - \hat{y}\|_0 < \varepsilon$, then $J[\hat{y}]$ is certainly an extremum with respect to all y such that $\|y - \hat{y}\|_1 < \varepsilon$. However, the converse is not true in general, i.e., a weak extremum may not be a strong extremum. As a rule, finding a weak extremum is simpler than finding a strong extremum. The reason for this is that the functionals usually considered in the calculus of variations are continuous in the norm of the space $\mathscr{D}_1$ (as noted at the end of the previous section), and this continuity can be exploited in the theory of weak extrema. In general, however, our functionals will not be continuous in the norm of the space $\mathscr{C}$.

THEOREM 2. *A necessary condition for the differentiable functional $J[y]$ to have an extremum for $y = \hat{y}$ is that its variation vanish for $y = \hat{y}$, i.e., that*

$$\delta J[h] = 0$$

for $y = \hat{y}$ and all admissible h.

Proof. To be explicit, suppose $J[y]$ has a minimum for $y = \hat{y}$. According to the definition of the variation $\delta J[h]$, we have

$$\Delta J[h] = \delta J[h] + \varepsilon\|h\|, \tag{9}$$

where $\varepsilon \to 0$ as $\|h\| \to 0$. Thus, for sufficiently small $\|h\|$, the sign of $\Delta J[h]$ will be the same as the sign of $\delta J[h]$. Now, suppose that $\delta J[h_0] \neq 0$ for some admissible h_0. Then for any $\alpha > 0$, no matter how small, we have

$$\delta J[-\alpha h_0] = -\delta J[\alpha h_0].$$

Hence, (9) can be made to have either sign for arbitrarily small $\|h\|$. But this is impossible, since by hypothesis $J[y]$ has a minimum for $y = \hat{y}$, i.e.,

$$\Delta J[h] = J[\hat{y} + h] - J[\hat{y}] \geqslant 0$$

for all sufficiently small $\|h\|$. This contradiction proves the theorem.

Remark. In elementary analysis, it is proved that for a function to have a minimum, it is necessary not only that its first differential vanish ($df = 0$), but also that its second differential be nonnegative. Consideration of the analogous problem for functionals will be postponed until Chapter 5.

4. The Simplest Variational Problem. Euler's Equation

4.1. We begin our study of concrete variational problems by considering what might be called the "simplest" variational problem, which can be formulated as follows: *Let* $F(x, y, z)$ *be a function with continuous first and second (partial) derivatives with respect to all its arguments. Then, among all functions* $y(x)$ *which are continuously differentiable for* $a \leqslant x \leqslant b$ *and satisfy the boundary conditions*

$$y(a) = A, \qquad y(b) = B, \tag{10}$$

find the function for which the functional

$$J[y] = \int_a^b F(x, y, y')\,dx \tag{11}$$

has a weak extremum. In other words, the simplest variational problem consists of finding a weak extremum of a functional of the form (11), where the class of admissible curves (see p. 8) consists of all smooth curves joining two points. The first two examples on pp. 2, 3, involving the brachistochrone and the shortest distance between two points, are variational problems of just this type. To apply the necessary condition for an extremum (found in Sec. 3.2) to the problem just formulated, we have to be able to calculate the variation of a functional of the type (11). We now derive the appropriate formula for this variation.

Suppose we give $y(x)$ an increment $h(x)$, where, in order for the function

$$y(x) + h(x)$$

to continue to satisfy the boundary conditions, we must have

$$h(a) = h(b) = 0.$$

Then, since the corresponding increment of the functional (11) equals

$$\begin{aligned}\Delta J = J[y + h] - J[y] &= \int_a^b F(x, y + h, y' + h')\,dx - \int_a^b F(x, y, y')\,dx \\ &= \int_a^b [F(x, y + h, y' + h') - F(x, y, y')]\,dx,\end{aligned}$$

it follows by using Taylor's theorem that

$$\Delta J = \int_a^b [F_y(x, y, y')h + F_{y'}(x, y, y')h']\,dx + \cdots, \tag{12}$$

where the subscripts denote partial derivatives with respect to the corresponding arguments, and the dots denote terms of order higher than 1 relative to h and h'. The integral in the right-hand side of (12) represents the principal linear part of the increment ΔJ, and hence the variation of $J[y]$ is

$$\delta J = \int_a^b [F_y(x, y, y')h + F_{y'}(x, y, y')h']\, dx.$$

According to Theorem 2 of Sec. 3.2, a necessary condition for $J[y]$ to have an extremum for $y = y(x)$ is that

$$\delta J = \int_a^b (F_y h + F_{y'} h')\, dx = 0 \tag{13}$$

for all admissible h. But according to Lemma 4 of Sec. 3.1, (13) implies that

$$F_y - \frac{d}{dx} F_{y'} = 0, \tag{14}$$

a result known as *Euler's equation.*[7] Thus, we have proved

THEOREM 1. *Let $J[y]$ be a functional of the form*

$$\int_a^b F(x, y, y')\, dx,$$

defined on the set of functions $y(x)$ which have continuous first derivatives in $[a, b]$ and satisfy the boundary conditions $y(a) = A$, $y(b) = B$. Then a necessary condition for $J[y]$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfy Euler's equation[8]

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

The integral curves of Euler's equation are called *extremals.* Since Euler's equation is a second-order differential equation, its solution will in general depend on two arbitrary constants, which are determined from the boundary conditions $y(a) = A$, $y(b) = B$. The problem usually considered in the theory of differential equations is that of finding a solution which is defined in the neighborhood of some point and satisfies given initial conditions (*Cauchy's problem*). However, in solving Euler's equation, we are looking for a solution which is defined over all of some fixed region and satisfies given boundary conditions. Therefore, the question of whether or not a certain variational problem has a solution does not just reduce to the

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[7] We emphasize that the existence of the derivative $(d/dx)F_{y'}$ is not assumed in advance, but follows from the very same lemma.

[8] This condition is necessary for a weak extremum. Since every strong extremum is simultaneously a weak extremum, any necessary condition for a weak extremum is also a necessary condition for a strong extremum.

usual existence theorems for differential equations. In this regard, we now state a theorem due to Bernstein,[9] concerning the existence and uniqueness of solutions "in the large" of an equation of the form

$$y'' = F(x, y, y'). \tag{15}$$

THEOREM 2 (*Bernstein*). *If the functions F, F_y and $F_{y'}$ are continuous at every finite point (x, y) for any finite y', and if a constant $k > 0$ and functions*

$$\alpha = \alpha(x, y) \geqslant 0, \qquad \beta = \beta(x, y) \geqslant 0$$

(which are bounded in every finite region of the plane) can be found such that

$$F_y(x, y, y') > k, \qquad |F(x, y, y')| \leqslant \alpha y'^2 + \beta,$$

then one and only one integral curve of equation (15) passes through any two points (a, A) and (b, B) with different abscissas $(a \neq b)$.

Equation (13) gives a necessary condition for an extremum, but in general, one which is not sufficient. The question of sufficient conditions for an extremum will be considered in Chapter 5. In many cases, however, Euler's equation by itself is enough to give a complete solution of the problem. In fact, the existence of an extremum is often clear from the physical or geometric meaning of the problem, e.g., in the brachistochrone problem, the problem concerning the shortest distance between two points, etc. If in such a case there exists only one extremal satisfying the boundary conditions of the problem, this extremal must perforce be the curve for which the extremum is achieved.

For a functional of the form

$$\int_a^b F(x, y, y')\, dx$$

Euler's equation is in general a second-order differential equation, but it may turn out that the curve for which the functional has its extremum is not twice differentiable. For example, consider the functional

$$J[y] = \int_{-1}^{1} y^2(2x - y')^2\, dx,$$

where

$$y(-1) = 0, \qquad y(1) = 1.$$

The minimum of $J[y]$ equals zero and is achieved for the function

$$y = y(x) = \begin{cases} 0 & \text{for} \quad -1 \leqslant x \leqslant 0, \\ x^2 & \text{for} \quad 0 < x \leqslant 1, \end{cases}$$

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[9] S. N. Bernstein, *Sur les équations du calcul des variations*, Ann. Sci. École Norm. Sup., **29**, 431–485 (1912).

which has no second derivative for $x = 0$. Nevertheless, $y(x)$ satisfies the appropriate Euler equation. In fact, since in this case

$$F(x, y, y') = y^2(2x - y')^2,$$

it follows that all the functions

$$F_y = 2y(2x - y')^2, \quad F_{y'} = -2y^2(2x - y'), \quad \frac{d}{dx} F_{y'}$$

vanish identically for $-1 \leqslant x \leqslant 1$. Thus, despite the fact that Euler's equation is of the second order and $y''(x)$ does not exist everywhere in $[-1, 1]$, substitution of $y(x)$ into Euler's equation converts it into an identity.

We now give conditions guaranteeing that a solution of Euler's equation has a second derivative:

THEOREM 3. *Suppose $y = y(x)$ has a continuous first derivative and satisfies Euler's equation*

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

Then, if the function $F(x, y, y')$ has continuous first and second derivatives with respect to all its arguments, $y(x)$ has a continuous second derivative at all points (x, y) where

$$F_{y'y'}[x, y(x), y'(x)] \neq 0.$$

Proof. Consider the difference

$$\begin{aligned} \Delta F_{y'} &= F_{y'}(x + \Delta x, y + \Delta y, y' + \Delta y') - F_{y'}(x, y, y') \\ &= \Delta x \bar{F}_{y'x} + \Delta y \bar{F}_{y'y} + \Delta y' \bar{F}_{y'y'}, \end{aligned}$$

where the overbar indicates that the corresponding derivatives are evaluated along certain intermediate curves. We divide this difference by Δx, and consider the limit of the resulting expression

$$\bar{F}_{y'x} + \frac{\Delta y}{\Delta x} \bar{F}_{y'y} + \frac{\Delta y'}{\Delta x} \bar{F}_{y'y'}$$

as $\Delta x \to 0$. (This limit exists, since $F_{y'}$ has a derivative with respect to x, which, according to Euler's equation, equals F_y.) Since, by hypothesis, the second derivatives of $F(x, y, z)$ are continuous, then, as $\Delta x \to 0$, $\bar{F}_{y'x}$ converges to $F_{y'x}$, i.e., to the value of $\partial^2 F / \partial y' \, \partial x$ at the point x. It follows from the existence of y' and the continuity of the second derivative $F_{y'y}$ that the second term $(\Delta y / \Delta x) \bar{F}_{y'y}$ also has a limit as $\Delta x \to 0$. But then the third term also has a limit (since the limit of the sum of the three terms exists), i.e., the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y'}{\Delta x} \bar{F}_{y'y'}$$

exists. As $\Delta x \to 0$, $\bar{F}_{y'y'}$ converges to $F_{y'y'} \neq 0$, and hence

$$\lim_{\Delta x \to 0} \frac{\Delta y'}{\Delta x} = y''(x)$$

exists. Finally, from the equation

$$\frac{d}{dx} F_{y'} - F_y = 0,$$

we can find an expression for y'', from which it is clear that y'' is continuous wherever $F_{y'y'} \neq 0$. This proves the theorem.

Remark. Here it is assumed that the extremals are *smooth.*[10] In Sec. 15 we shall consider the case where the solution of a variational problem may only be *piecewise smooth*, i.e., may have "corners" at certain points.

4.2. Euler's equation (14) plays a fundamental role in the calculus of variations, and is in general a second-order differential equation. We now indicate some special cases where Euler's equation can be reduced to a first-order differential equation, or where its solution can be obtained entirely in terms of quadratures (i.e., by evaluating integrals).

Case 1. Suppose the integrand does not depend on y, i.e., let the functional under consideration have the form

$$\int_a^b F(x, y')\, dx,$$

where F does not contain y explicitly. In this case, Euler's equation becomes

$$\frac{d}{dx} F_{y'} = 0,$$

which obviously has the first integral

$$F_{y'} = C, \tag{16}$$

where C is a constant. This is a first-order differential equation which does not contain y. Solving (16) for y', we obtain an equation of the form

$$y' = f(x, C),$$

from which y can be found by a quadrature.

Case 2. If the integrand does not depend on x, i.e., if

$$J[y] = \int_a^b F(y, y')\, dx,$$

then

$$F_y - \frac{d}{dx} F_{y'} = F_y - F_{y'y} y' - F_{y'y'} y''. \tag{17}$$

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[10] We say that the function $y(x)$ is *smooth* in an interval $[a, b]$ if it is continuous in $[a, b]$, and has a continuous derivative in $[a, b]$. We say that $y(x)$ is *piecewise smooth* in $[a, b]$ if it is continuous everywhere in $[a, b]$, and has a continuous derivative in $[a, b]$ except possibly at a finite number of points.

Multiplying (17) by y', we obtain

$$F_y y' - F_{y'y} y'^2 - F_{y'y'} y' y'' = \frac{d}{dx}(F - y' F_{y'}).$$

Thus, in this case, Euler's equation has the first integral

$$F - y' F_{y'} = C,$$

where C is a constant.

Case 3. If F does not depend on y', Euler's equation takes the form

$$F_y(x, y) = 0,$$

and hence is not a differential equation, but a "finite" equation, whose solution consists of one or more curves $y = y(x)$.

Case 4. In a variety of problems, one encounters functionals of the form

$$\int_a^b f(x, y)\sqrt{1 + y'^2}\, dx,$$

representing the integral of a function $f(x, y)$ with respect to the *arc length* $s\,(ds = \sqrt{1 + y'^2}\, dx)$. In this case, Euler's equation can be transformed into

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) &= f_y(x, y)\sqrt{1 + y'^2} - \frac{d}{dx}\left[f(x, y)\frac{y'}{\sqrt{1 + y'^2}}\right] \\ &= f_y\sqrt{1 + y'^2} - f_x\frac{y'}{\sqrt{1 + y'^2}} - f_y\frac{y'^2}{\sqrt{1 + y'^2}} - f\frac{y''}{(1 + y'^2)^{3/2}} \\ &= \frac{1}{\sqrt{1 + y'^2}}\left[f_y - f_x y' - f\frac{y''}{1 + y'^2}\right] = 0, \end{aligned}$$

i.e.,

$$f_y - f_x y' - f\frac{y''}{1 + y'^2} = 0.$$

Example 1. Suppose that

$$J[y] = \int_1^2 \frac{\sqrt{1 + y'^2}}{x}\, dx, \quad y(1) = 0, \quad y(2) = 1.$$

The integrand does not contain y, and hence Euler's equation has the form $F_{y'} = C$ (cf. Case 1). Thus,

$$\frac{y'}{x\sqrt{1 + y'^2}} = C,$$

so that

$$y'^2(1 - C^2x^2) = C^2x^2$$

or

$$y' = \frac{Cx}{\sqrt{1 - C^2x^2}},$$

from which it follows that

$$y = \int \frac{Cx\,dx}{\sqrt{1 - C^2x^2}} = \frac{1}{C}\sqrt{1 - C^2x^2} + C_1$$

or

$$(y - C_1)^2 + x^2 = \frac{1}{C^2}.$$

Thus, the solution is a circle with its center on the y-axis. From the conditions $y(1) = 0$, $y(2) = 1$, we find that

$$C = \frac{1}{\sqrt{5}}, \qquad C_1 = 2,$$

so that the final solution is

$$(y - 2)^2 + x^2 = 5.$$

Example 2. *Among all the curves joining two given points* (x_0, y_0) *and* (x_1, y_1), *find the one which generates the surface of minimum area when rotated about the x-axis.* As we know, the area of the surface of revolution generated by rotating the curve $y = y(x)$ about the x-axis is

$$2\pi \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\,dx.$$

Since the integrand does not depend explicitly on x, Euler's equation has the first integral

$$F - y'F_{y'} = C$$

(cf. Case 2), i.e.,

$$y\sqrt{1 + y'^2} - y\frac{y'^2}{\sqrt{1 + y'^2}} = C$$

or

$$y = C\sqrt{1 + y'^2},$$

so that

$$y' = \sqrt{\frac{y^2 - C^2}{C^2}}.$$

Separating variables, we obtain

$$dx = \frac{C\,dy}{\sqrt{y^2 - C^2}},$$

i.e.,

$$x + C_1 = C \ln \frac{y + \sqrt{y^2 - C^2}}{C},$$

so that

$$y = C \cosh \frac{x + C_1}{C}. \tag{18}$$

Thus, the required curve is a *catenary* passing through the two given points. The surface generated by rotation of the catenary is called a *catenoid.* The values of the arbitrary constants C and C_1 are determined by the conditions

$$y(x_0) = y_0, \qquad y(x_1) = y_1.$$

It can be shown that the following three cases are possible, depending on the positions of the points (x_0, y_0) and (x_1, y_1):

1. If a single curve of the form (18) can be drawn through the points (x_0, y_0) and (x_1, y_1), this curve is the solution of the problem [see Figure 2(a)].
2. If two extremals can be drawn through the points (x_0, y_0) and (x_1, y_1), one of the curves actually corresponds to the surface of revolution of minimum area, and the other does not.
3. If there is no curve of the form (18) passing through the points (x_0, y_0) and (x_1, y_1), there is no surface in the class of smooth surfaces of revolution which achieves the minimum area. In fact, if the location of the two points is such that the distance between them is sufficiently large compared to their distances from the x-axis, then the area of the surface consisting of two circles of radius y_0 and y_1, plus the segment of the x-axis joining them [see Figure 2(b)] will be less than the area of any surface of revolution generated by a smooth curve passing through the points. Thus, in this case the surface of revolution generated by the polygonal line Ax_0x_1B has the minimum area, and there is no surface of minimum area in the class of surfaces generated by rotation about the x-axis of smooth curves passing through the given points. (This case, corresponding to a "broken extremal," will be discussed further in Sec. 15.)

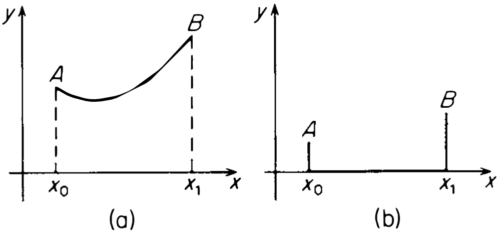

FIGURE 2

Example 3. For the functional

$$J[y] = \int_a^b (x - y)^2\, dx, \tag{19}$$

Euler's equation reduces to a finite equation (see Case 3), whose solution is the straight line $y = x$. In fact, the integral (19) vanishes along this line.

5. The Case of Several Variables

So far, we have considered functionals depending on functions of one variable, i.e., on curves. In many problems, however, one encounters functionals depending on functions of several independent variables, i.e., on surfaces. Such multidimensional problems will be considered in detail in Chapter 7. For the time being, we merely give an idea of how the formulation and solution of the simplest variational problem discussed above carries over to the case of functionals depending on surfaces.

To keep the notation simple, we confine ourselves to the case of two independent variables, but all our considerations remain the same when there are n independent variables. Thus, let $F(x, y, z, p, q)$ be a function with continuous first and second (partial) derivatives with respect to all its arguments, and consider a functional of the form

$$J[z] = \iint_R F(x, y, z, z_x, z_y)\,dx\,dy, \tag{20}$$

where R is some closed region and z_x, z_y are the partial derivatives of $z = z(x, y)$. Suppose we are looking for a function $z(x, y)$ such that

1. $z(x, y)$ and its first and second derivatives are continuous in R;
2. $z(x, y)$ takes given values on the boundary Γ of R;
3. The functional (20) has an extremum for $z = z(x, y)$.

Since the proof of Theorem 2 of Sec. 3.2 does not depend on the form of the functional J, then, just as in the case of one variable, a necessary condition for the functional (20) to have an extremum is that its variation (i.e., the principal linear part of its increment) vanish. However, to find Euler's equation for the functional (20), we need the following lemma, which is analogous to Lemma 1 of Sec. 3.1 (see also the remark on p. 9):

LEMMA. *If $\alpha(x, y)$ is a fixed function which is continuous in a closed region R, and if the integral*

$$\iint_R \alpha(x, y)h(x, y)\,dx\,dy \tag{21}$$

vanishes for every function $h(x, y)$ which has continuous first and second derivatives in R and equals zero on the boundary Γ of R, then $\alpha(x, y) = 0$ everywhere in R.

Proof. Suppose the function $\alpha(x, y)$ is nonzero, say positive, at some point in R. Then $\alpha(x, y)$ is also positive in some circle

$$(x - x_0)^2 + (y - y_0)^2 \leqslant \varepsilon^2 \tag{22}$$

6. A Simple Variable End Point Problem

There are, of course, many other kinds of variational problems besides the "simplest" variational problem considered so far, and such problems will be studied in Chapters 2 and 3. However, this is a suitable place for acquainting the reader with one of these problems, i.e., the *variable end point problem*, a particular case of which can be stated as follows: *Among all curves whose end points lie on two given vertical lines* $x = a$ *and* $x = b$, *find the curve for which the functional*

$$J[y] = \int_a^b F(x, y, y')\, dx \tag{26}$$

has an extremum.[13]

We begin by calculating the variation δJ of the functional (26). As before, δJ means the principal linear part of the increment

$$\Delta J = J[y + h] - J[y] = \int_a^b [F(x, y + h, y' + h') - F(x, y, y')]\, dx.$$

Using Taylor's theorem to expand the integrand, we obtain

$$\Delta J = \int_a^b (F_y h + F_{y'} h')\, dx + \cdots,$$

where the dots denote terms of order higher than 1 relative to h and h', and hence

$$\delta J = \int_a^b (F_y h + F_{y'} h')\, dx.$$

Here, unlike the fixed end point problem, $h(x)$ need no longer vanish at the points a and b, so that integration by parts now gives[14]

$$\begin{aligned} \delta J &= \int_a^b \left(F_y - \frac{d}{dx} F_{y'}\right) h(x)\, dx + F_{y'} h(x)\Big|_{x=a}^{x=b} \\ &= \int_a^b \left(F_y - \frac{d}{dx} F_{y'}\right) h(x)\, dx + F_{y'}\big|_{x=b}\, h(b) - F_{y'}\big|_{x=a}\, h(a). \end{aligned} \tag{27}$$

We first consider functions $h(x)$ such that $h(a) = h(b) = 0$. Then, as in the simplest variational problem, the condition $\delta J = 0$ implies that

$$F_y - \frac{d}{dx} F_{y'} = 0. \tag{28}$$

Therefore, in order for the curve $y = y(x)$ to be a solution of the variable end point problem, y must be an extremal, i.e., a solution of Euler's equation.

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[13] The more general case where the end points lie on two given curves $y = \varphi(x)$ and $y = \psi(x)$ is treated in Sec. 14.

[14] As usual, $f(x)\big|_{x=a}^{x=b}$ stands for $f(b) - f(a)$.

But if y is an extremal, the integral in the expression (27) for δJ vanishes, and then the condition $\delta J = 0$ takes the form

$$F_{y'}|_{x=b}\, h(b) - F_{y'}|_{x=a}\, h(a) = 0,$$

from which it follows that

$$F_{y'}|_{x=a} = 0, \qquad F_{y'}|_{x=b} = 0, \tag{29}$$

since $h(x)$ is arbitrary. Thus, to solve the variable end point problem, we must first find a general integral of Euler's equation (28), and then use the conditions (29), sometimes called the *natural boundary conditions*, to determine the values of the arbitrary constants.

Besides the case of fixed end points and the case of variable end points, we can also consider the *mixed case*, where one end is fixed and the other is variable. For example, suppose we are looking for an extremum of the functional (26) with respect to the class of curves joining a given point A (with abscissa a) and an arbitrary point of the line $x = b$. In this case, the conditions (29) reduce to the single condition

$$F_{y'}|_{x=b} = 0,$$

and $y(a) = A$ serves as the second boundary condition.

Example. *Starting from the point $P = (a, A)$, a heavy particle slides down a curve in the vertical plane. Find the curve such that the particle reaches the vertical line $x = b$ $(\neq a)$ in the shortest time.* (This is a variant of the brachistochrone problem, p. 3.)

For simplicity, we assume that the original point coincides with the origin of coordinates. Since the velocity of motion along the curve equals

$$v = \frac{ds}{dt} = \sqrt{1 + y'^2}\,\frac{dx}{dt},$$

we have

$$dt = \frac{\sqrt{1 + y'^2}}{v}\,dx = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\,dx,$$

so that the transit time T is given by the equation

$$T = \int \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\,dx.$$

The general solution of the corresponding Euler equation consists of a family of cycloids

$$x = r(\theta - \sin\theta) + c, \qquad y = r(1 - \cos\theta).$$

Since the curve must pass through the origin, we must have $c = 0$. To determine r, we use the second condition

$$F_{y'} = \frac{y'}{\sqrt{2gy}\sqrt{1 + y'^2}} = 0 \qquad \text{for} \quad x = b,$$

i.e., $y' = 0$ for $x = b$, which means that the tangent to the curve at its right end point must be horizontal. It follows that $r = b/\pi$, and hence the required curve is given by the equations

$$x = \frac{b}{\pi}(\theta - \sin\theta), \qquad y = \frac{b}{\pi}(1 - \cos\theta).$$

7. The Variational Derivative

In Sec. 3.2 we introduced the concept of the differential of a functional. We now introduce the concept of the *variational* (or *functional*) *derivative*, which plays the same role for functionals as the concept of the partial derivative plays for functions of n variables. We begin by considering functionals of the type

$$J[y] = \int_a^b F(x, y, y')\,dx, \qquad y(a) = A, \quad y(b) = B, \tag{30}$$

corresponding to the simplest variational problem. Our approach is to first go from the variational problem to an n-dimensional problem, and then pass to the limit $n \to \infty$.

Thus, we divide the interval $[a, b]$ into $n + 1$ equal subintervals by introducing the points

$$x_0 = a, \; x_1, \ldots, x_n, \; x_{n+1} = b, \qquad (x_{i+1} - x_i = \Delta x),$$

and we replace the smooth function $y(x)$ by the polygonal line with vertices

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1}),$$

where $y_i = y(x_i)$.[15] Then (30) can be approximated by the sum

$$J(y_1, \ldots, y_n) \equiv \sum_{i=0}^{n} F\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right)\Delta x, \tag{31}$$

which is a function of n variables. (Recall that $y_0 = A$ and $y_{n+1} = B$ are fixed.)

Next, we calculate the partial derivatives

$$\frac{\partial J(y_1, \ldots, y_n)}{\partial y_k},$$

and we consider what happens to these derivatives as the number of points of subdivision increases without limit. Observing that each variable y_k

---

[15] This is the *method of finite differences* (cf. Secs. 1, 40).

AN INTRODUCTION TO LINEAR PROGRAMMING AND GAME THEORY

Third Edition

Paul R. Thie

G. E. Keough

WILEY

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CHAPTER 2

THE LINEAR PROGRAMMING MODEL

2.1 HISTORY

The basic problem of linear programming, determining the optimal value of a linear function subject to linear constraints, arises in a wide variety of situations, but the theory that we will develop is of recent origin.

In 1939 the Russian mathematician L. V. Kantorovich published a monograph entitled Mathematical Methods in the Organization and Planning of Production [2]. Kantorovich recognized that a broad class of production problems led to the same mathematical problem and that this problem was susceptible to solution by numerical methods. However, Kantorovich's work went unrecognized.

In 1941 Frank Hitchcock [3] formulated the transportation problem, and in 1945 George Stigler [1] considered the problem referred to in Section 1.2 of determining an adequate diet for an individual at minimal cost. Through these problems and others, especially problems related to the World War II effort, it became clear that a feasible method for solving linear programming problems was needed. Then in 1951 George Dantzig [4] developed the simplex method. This technique is the basis of the next chapter. John von Neumann recognized the importance of the concept of duality, the mathematical thread uniting linear programming and game theory, and the first published proof of the Duality Theorem is that of Gale, Kuhn, and Tucker [5].

Since the late 1940s, many other computational techniques and variations have been devised, usually for specific types of problems or for use with certain types of computing hardware. The theory has been applied extensively in industry. On the one hand, management has been forced to define explicitly its desired objectives and given constraints. This has brought about a much greater understanding of the decision-making process. On the other hand, the actual techniques of linear programming have been successfully applied in the petroleum industry, the food processing industry, the iron and steel industry, and many more.

Theoretical developments in linear programming have attracted the attention of both theoreticians and the practitioners in the field (along with the readers of the *New York Times*). Some comments on these events are included in Appendix C on theory and efficiency in linear programming

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2.2 THE BLENDING MODEL

The diet problem described in Section 1.2 is an example of a general type of linear programming problem that involves blending or combining various ingredients. The cost and composition or characteristics of the various ingredients are known, and the problem is to determine how much of each of the ingredients to blend together so that the total cost of the mixture is minimized while the composition of the mixture satisfies specified requirements. In the diet problem, foods were combined to form a diet minimizing costs and meeting basic nutritional requirements.

The construction of the mathematical model for problems of this type follows quickly once the usually more difficult task of defining the characteristics and cost of the ingredients and required composition of the blend has been accomplished. Assuming that all this information is at hand, the amounts of each of the ingredients to blend together must be decided. Thus, variables are assigned to represent these amounts. The cost function, the function to be optimized, can then be constructed by considering the cost of each of the ingredients and assuming that the total cost is the sum of the individual costs. The system of constraints, that is, the set of restrictions of the variables, follows by considering the requirements specified for the final blend.

Example 2.2.1. To feed her stock a farmer can purchase two kinds of feed. The farmer has determined that the herd requires 60, 84, and 72 units of the nutritional elements A, B, and C, respectively, per day. The contents and cost of a pound of each of the two feeds are given in the following table.

	Nutritional Elements (units/lb)			
	A	*B*	*C*	*Cost (cents/lb)*
Feed 1	3	7	3	10
Feed 2	2	2	6	4

Obviously, the farmer could use only one feed to meet the daily nutritional requirements. For example, it can easily be seen that 24 lb of the first feed would provide an adequate diet at a daily cost of \$2.40. However, the farmer wants to determine the least expensive way of providing an adequate diet by combining the two feeds. To do this, the farmer should consider all possible diets that satisfy the specified requirements and then select from this set the diet of minimal cost.

To translate this into a mathematical problem, let x be the number of pounds of Feed 1 and y the number of pounds of Feed 2 to be used in the daily diet. Then by definition, x and y must be nonnegative. Moreover, a diet consisting of x lb of Feed 1 and y lb of Feed 2 would contain $3x+2y$ units of nutritional element A. Since 60 units of element A are required daily, we must have $3x+2y \geq 60$. We are assuming that providing more than the minimal requirements of any of the nutritional elements will have no harmful effects, and so any diet providing at least 60 units of element A will satisfy this requirement. Thus the inequality and not an equality.

To provide insight into the nature of linear programming, this particular problem will be solved geometrically. The set of diets satisfying the above requirements can be illustrated graphically. All the points (x,y) in the first quadrant satisfying the inequality are shown in Figure 2.1.

The other two nutritional requirements demand that

$$7x + 2y \geq 84 \text{ and } 3x + 6y \geq 72$$

The corresponding regions in the first quadrant are sketched in Figure 2.2.

We must consider all feasible diets, that is, all diets that satisfy all three requirements. They are given graphically by the shaded region in Figure 2.3.

The cost in cents of a diet of x lb of Feed 1 and y lb of Feed 2 is $10x + 4y$. Thus we must determine the minimum of the function $f(x,y) = 10x + 4y$, while the x and y are restricted to the shaded region in Figure 2.3.

Consider the graphs of the family of lines determined by the equation $10x + 4y = c$, where c is constant. In Figure 2.4, some of these lines are graphed for various values of c. Note that all the lines have the same slope and that the lines move to the left as c decreases.

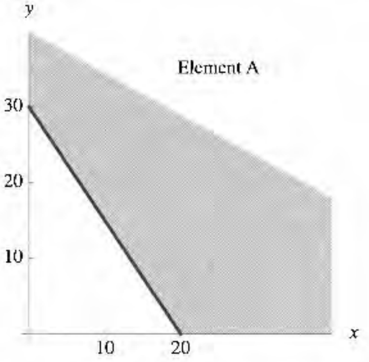

Figure 2.1

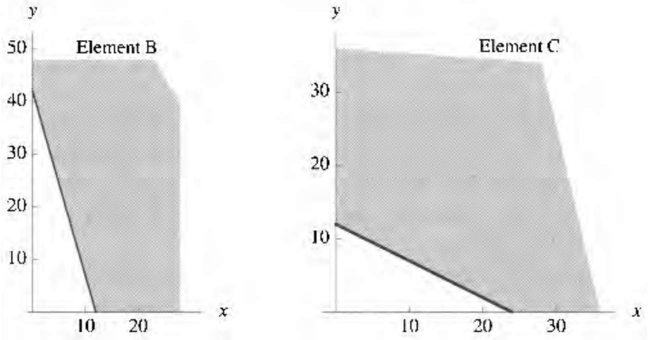

Figure 2.2

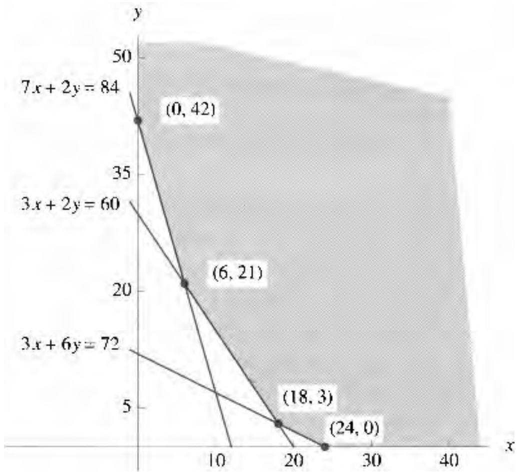

Figure 2.3

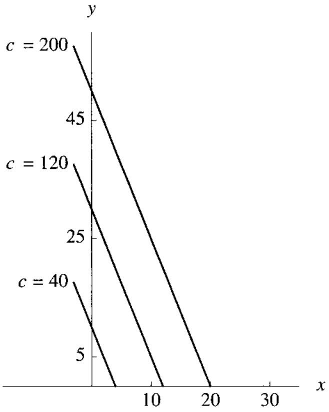

Figure 2.4

Each of the parallel lines consists of points that give the same value for the cost function $10x + 4y$. Thus we seek that line farthest to the left that still intersects the shaded region of Figure 2.3. The line through point $(6, 21)$ is that line, as illustrated in Figure 2.5. Thus the cost of a minimal diet is $10 \cdot 6 + 4 \cdot 21 = 144$ cents, and this diet consists of 6 lb of Feed 1 and 2 lb of Feed 2.

This analysis can be extended. As the value of c in the family of lines $10x + 4y = c$ decreases and the lines slide down and to the left, from the geometry it follows that the line we seek will intersect the set of feasible solutions at a corner point (or vertex) of the set of feasible solutions. In this example we can therefore conclude that a minimal-cost diet, if it exists, must be attained at either point $(0, 42)$, $(6, 21)$, $(18, 3)$,

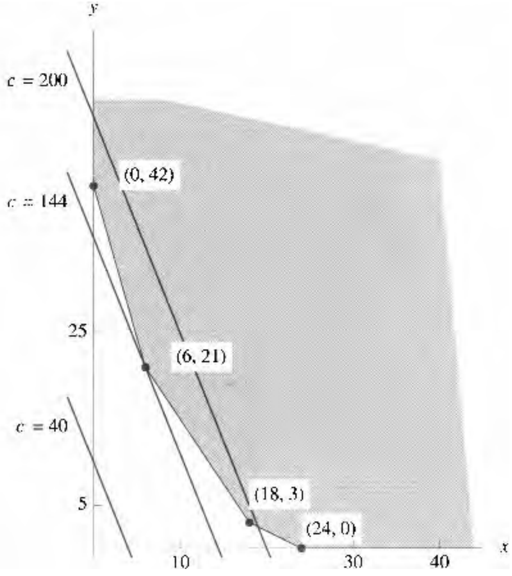

Figure 2.5

or $(24,0)$. Thus, if we have the corner points at hand, evaluating the cost function at each of these points and comparing values will yield the desired optimal diet:

corner points	$(0,42)$	$(6,21)$	$(18,3)$	$(24,0)$
$10x+4y$	168	144	192	240
		$\Uparrow$		

Our above result is confirmed; the minimal-cost diet is to use daily 6 lb of Feed 1 and 21 lb of Feed 2 at a cost of 144 cents.

Suppose now that the price of Feed 1 increases from 10 cents/lb to 14 cents/lb, with all other data unchanged. Then the corner points of the set of feasible solutions is as above, and an evaluation of the new cost function at these points will yield the revised optimal solution.

corner points	$(0,42)$	$(6,21)$	$(18,3)$	$(24,0)$
$14x+4y$	168	168	264	336
	$\Uparrow$	$\Uparrow$		

Now the optimal diet is not unique. The minimal-cost line $14x+4y=168$ passes through the two corner points $(0,42)$ and $(6,21)$, and since any feasible point on this line delivers a diet of 168 cents/lb, the set of optimal feasible diets consists of the points on the line segment between the corner points $(0,42)$ and $(6,21)$, as displayed in Figure 2.6.

We have in the solution to the above problem a function with a unique minimum value (certainly there can be only one minimum value) but with multiple optimal solution points. And in the example, with only two variables, the geometry justifies

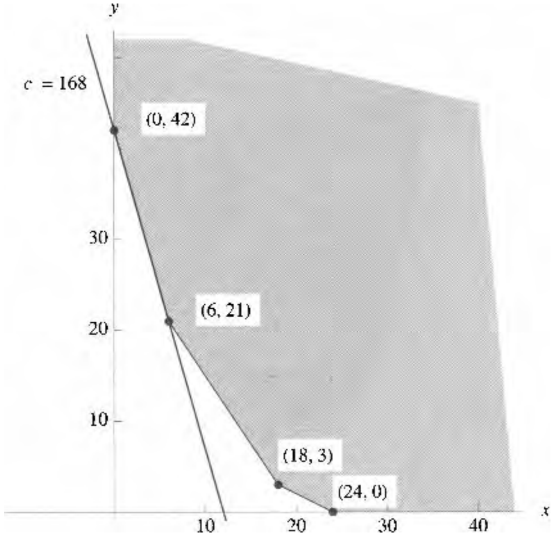

Figure 2.6

the result. The lines in the family $\{14x+4y=c : c \text{ a constant}\}$ and the boundary line $7x+2y=84$ are parallel, with common slope$-\frac{7}{2}$, and when c decreases, the line with a minimum value for c that intersects the set of feasible solutions will lie on the segment of the boundary corresponding to this constraining line.

The use of slopes can be extended. Consider the original cost function $10x+4y$. The slope of the associated family of lines $\{10x+4y=c : c \text{ a constant}\}$ is $-\frac{5}{2}$, and the optimal solution point to the problem, $(6,21)$, is at the intersection of the boundary lines $7x+2y=84$ (with slope$-\frac{7}{2}$) and $3x+2y=60$ (with slope$-\frac{3}{2}$). Thus from the geometry, the slope $-\frac{5}{2}$ of the function to be minimized must be between these two slopes. Indeed, $-\frac{7}{2}<-\frac{5}{2}<-\frac{3}{2}$.

In fact, we can say that if the cost function is c_1x+c_2y, where c_1 and c_2 are positive numbers, the minimum cost would be attained at the point $(6,21)$ if $-\frac{7}{2}\leq-\frac{c_1}{c_2}\leq-\frac{3}{2}$, that is, $\frac{3}{2}\leq\frac{c_1}{c_2}\leq\frac{7}{2}$, and the solution point would be unique if the inequalities are strict.

Thus, for example, if the cost c_2 of Feed 2 is fixed at 4 cents/lb but the cost c_1 of Feed 1 is variable, the farmer should continue to use the $(6,21)$ diet as long as $\frac{3}{2}\leq\frac{c_1}{4}\leq\frac{7}{2}$, that is, as long as $6\leq c_1\leq 14$, with a minimum daily cost of $6c_1+21\cdot 4=6c_1+84$ cents.

Example 2.2.2. A landscaper has on hand two grass seed blends. Blend I contains 60% bluegrass seed and 10% fescue and costs 80 cents/lb; Blend II contains 20% bluegrass seed and 50% fescue and costs 60 cents/lb. (Each also contains other types of seeds and inert materials.) The field about to be sowed requires a composition seed consisting of at least 30% bluegrass and 26% fescue. What is the least expensive combination of the two blends that meets these requirements?

To formulate a mathematical model for a problem involving percentages, ambiguities can arise. To avoid these, we can determine the optimal way to produce a fixed amount of the final product.

For example, let us determine the combination that minimizes costs and produces 100 lb of the required composition seed. Defining x as the number of pounds of Blend I used in this composition and y as the number of pounds of Blend II, the 30% bluegrass requirement translates into the inequality

$$0.60x + 0.20y \geq 30$$

as the 100 lb of the final composition must contain at least 30 lb of bluegrass. The fescue requirement yields the inequality

$$0.10x + 0.50y \geq 26$$

These inequalities simplify to $3x + y \geq 150$ and $x + 5y \geq 260$. The region in the first quadrant satisfying the inequalities is graphed in Figure 2.7.

Since 100 lb of the composition is to be produced, x and y must also satisfy the equation $x + y = 100$ (see Figure 2.8).

The cost in dollars of x lb of Blend I and y lb of Blend II is $c(x,y) = 0.8x + 0.6y$, and we seek the minimum of this linear function on the set of points represented by the heavy line in Figure 2.8. From the geometric argument of the previous example, it follows that the line in the family of parallel lines $\{(x,y) : 0.8x + 0.6y = c\}$, where c is a constant, with minimal c and intersecting this set must intersect the set at either $(25,75)$ or $(60,40)$. Evaluating,

$$c(25,75) = \$65 \text{ and } c(60,40) = \$72$$

Thus, to produce 100 lb of the composition at minimum cost, 25 lb of Blend I and 75 lb of Blend II should be used, and so the minimal-cost prescription for making any amount of the composition seed is to use 25% Blend I and 75% Blend II.

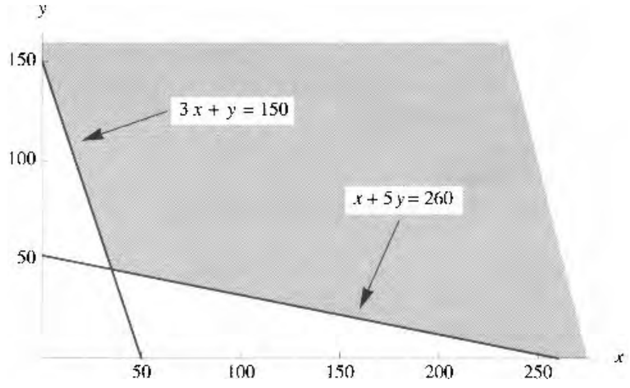

Figure 2.7

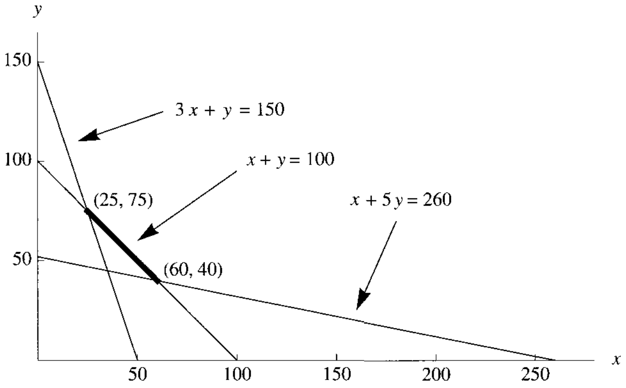

Figure 2.8

Example 2.2.3 (Continuation of Example 2.2.2). The operation of the landscaper of the above example has expanded. Now there are two fields to be maintained, Field X (the original field) and Field Y, with Field Y requiring a seed mixture that is at least 15% bluegrass and 35% fescue; and there is an additional grass seed blend to work with, Blend III, with a composition of 25% bluegrass and 15% fescue and a cost of 35 cents/lb. The relevant data are summarized in the following table.

		Bluegrass	*Fescue*	*Cost (cents/lb)*
	Blend I	60%	10%	80
Composition	*Blend II*	20%	50%	60
	Blend III	25%	15%	35
Requirements	*Field X*	$\geq 30\%$	$\geq 26\%$	
	Field Y	$\geq 15\%$	$\geq 35\%$	

Suppose the landscaper has an order for 100 lbs of seed for Field X and 160 lbs of seed for Field Y. To determine the minimum cost to meet these demands, the following model is formulated. Let x_1, x_2, x_3 be the number of pounds of Blends I, II, and III, respectively, used for Field X, and let y_1, y_2, y_3 be the number of pounds of each used for Field Y. The problem:

$$
\begin{array}{ll}
\text{To minimize the function} & \\
\quad (80x_1+60x_2+35x_3)+(80y_1+60y_2+35y_3) & \\
\text{subject to} & \\
x_1+x_2+x_3=100 & y_1+y_2+y_3=160 \\
.6x_1+.2x_2+.25x_3 \geq 30 & .6y_1+.2y_2+.25y_3 \geq .15(160)=24 \\
.1x_1+.5x_2+.15x_3 \geq 26 & .1y_1+.5y_2+.15y_3 \geq .35(160)=56 \\
x_1,x_2,x_3 \geq 0 & y_1,y_2,y_3 \geq 0
\end{array}
\tag{2.2.1}
$$

Unlike the optimization problems of Examples 2.2.1 and 2.2.2, each with only two variables, this problem, with six variables, cannot be solved graphically. The problems are essentially the same, with linear functions to be optimized subject to linear constraints. But any such problem with more than two variables is intractable to a graphical approach. The goal of Chapter 3 is to develop an efficient method of solving the general problem, regardless of size.

While we cannot complete problem (2.2.1) at this time, some further comments on the problem are in order. The reader may have already noted that (2.2.1) can be simplified. Meeting the demands for Field X and meeting the demands for Field Y are independent problems; the x's and the y's in (2.2.1) are not related in the family of constraints. We could solve each of these problems separately and then combine the solutions to resolve the two-field problem. (Of course, graphical solution techniques would remain out of reach for the two three-variable problems.)

On the other hand, further restrictions could easily eliminate this simplification. Suppose, for example, that only a limited amount of one of the blends is available — perhaps only 125 lbs of the new Blend III is on hand and can be used at this time. Then the constraint $x_3 + y_3 \leq 125$ would need to be added to (2.2.1), and the optimization problems for the two fields are no longer independent.

Another variation could be that, because of shipping restrictions, the producer of the seed can deliver Blends I and II only in a single drum containing a premixed combination of the two blends, with the customers specifying the ratio of Blend I to Blend II to be used in preparing their orders. In the landscaper model, this means that the ratios of Blend I to Blend II used in each of the fields are the same, that is, $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ or $x_1y_2 = x_2y_1$. However, adding the simple equality $x_1y_2 = x_2y_1$ to (2.2.1) changes the optimization problem dramatically. The problem is no longer a linear programming problem, as $x_1y_2 = x_2y_1$ is not a linear constraint. The problem is in the domain of nonlinear programming, a topic not considered in this linear programming text.

Problem Set 2.2

Problems 1–5 refer to Example 2.2.1.

1. A salesperson offers the farmer a new feed for her stock. One pound of this feed contains 2, 4, and 4 units of the nutritional elements A, B, and C, respectively, and costs 7 cents. By considering a blend that consists of equal parts of Feeds 1 and 2, show that the use of this new feed cannot reduce the minimal cost of an adequate diet.

2. The farmer has determined that as long as the ratio of the cost of Feed 1 to the cost of Feed 2 is between $\frac{1}{2}$ and $\frac{3}{2}$, an adequate diet of minimal cost can be achieved by using 18 lb of Feed 1 and 3 lb of Feed 2. Explain.

3. What should the ratio of the costs of the feeds be to warrant the use of a diet consisting solely of Feed 1? When should the farmer use only Feed 2 for her stock?

4. After reviewing his mother's mathematical formulation of the feed problem, the farmer's son claims that in general the constraining inequalities should be equal-

ities. He reasons that money must be wasted if some of the nutritional elements are fed to the stock at a level above the minimal requirements. Is this true?

5. After some study, the farmer has decided that 40 units of nutritional element D are also critical for the daily feeding of his stock. One pound of Feeds 1 and 2 contains 4 and 2 units of element D, respectively. How does this change the analysis of the original problem?

6. Products X and Y are to be blended to produce a mixture that is at least 30% A and 30% B. Product X is 50% A and 40% B and costs \$10/gal; Product Y is 20% A and 10% B and costs \$2/gal. To formulate a model to be used to determine a minimal-cost blend, we let x and y equal the number of gallons of X and Y used, respectively, and write the following mathematical problems:

   (a) Our first attempt.

   $$\begin{array}{l} \text{Minimize } 10x+2y \\ \text{subject to} \\ .5x + .2y \geq .3 \\ .4x + .1y \geq .3 \\ x,y \geq 0 \end{array}$$

   Note that $x = 0$, $y = 3$ satisfies the constraints. So should we use only Product Y? Explain.

   (b) We try again. Our final product is to be at least 30% A and 30% B and contain $x+y$ gal, so we want to

   $$\begin{array}{l} \text{Minimize } 10x+2y \\ \text{subject to} \\ .5x + .2y \geq .3(x+y) \\ .4x + .1y \geq .3(x+y) \\ x,y \geq 0 \end{array}$$

   But does $x = 0$, $y = 0$ satisfy the constraints? Explain.

   (c) Formulate a correct model.

For Problems 7–10, formulate mathematical models and then solve the problems.

7. (a) A poultry producer's stock requires at least 124 units of nutritional element A and 60 units of nutritional element B daily. Two feeds are available for use. One pound of Feed 1 costs 16 cents and contains 10 units of A and 3 units of B. One pound of Feed 2 costs 14 cents and contains 4 units of A and 5 units of B. Determine for the producer the least expensive adequate feeding diet.

   (b) For what range on the ratio of the costs of Feed 1 to Feed 2 would the optimal diet be the above diet?

   (c) For what values of the ratio of the costs of Feed 1 to Feed 2 would the optimal diet for the problem of part (a) not be unique?

CHAPTER 3

THE SIMPLEX METHOD

3.1 THE GENERAL PROBLEM

In the previous chapter, all examples led to one basic mathematical problem: the optimization of a linear function subject to a system of linear constraints. In this chapter we will develop a technique for solving this basic problem.

One minor complication in studying the problem is that the optimization problem can take various forms. For example, we have seen both maximization and minimization problems and constraint sets that have consisted of equalities and inequalities in both directions. However, this difficulty is easily resolved because all linear programming problems can be transformed into equivalent problems that are in what we call *standard form.*

Definition 3.1.1. The *standard form* of the linear programming problem is to determine a solution of a set of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{3.1.1}$$

with

$$x_j \geq 0, j = 1, \ldots, n$$

that minimizes the function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n - z_0$$

(The $-z_0$ term allows for the inclusion of a constant in the expression for the function to be optimized. In an application such a constant could represent, for example, fixed costs or guaranteed benefits. We precede the constant with a negative sign for future convenience; z_0 can be positive, negative, or zero.)

It is this standard form of the linear programming problem, a minimization problem involving only equalities, that we will solve. Thus our first task is to show that any linear programming problem can be formulated as a problem in standard form, where the number of equalities, m, and the number of variables, n, are determined by the problem.

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Consider first a linear programming problem with a system of constraints that contains inequalities. For example, suppose a particular diet problem reduces to the mathematical problem of minimizing $3x_1 + 2x_2 + 4x_3$ subject to the constraints

$$\begin{aligned} 30x_1 + 100x_2 + 85x_3 &\leq 2500 \\ 6x_1 + \quad 2x_2 + \quad 3x_3 &\geq \quad 90 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Such a problem could result from seeking minimal-cost diet that places an upper bound on calorie intake and a lower bound on protein intake. We will show that this problem is equivalent to the following problem derived from the original problem by the addition of two new nonnegative variables, x_4 and x_5.

$$\begin{aligned} &\text{Minimize } 3x_1 + 2x_2 + 4x_3 \\ &\text{subject to} \\ &30x_1 + 100x_2 + 85x_3 + x_4 \qquad = 2500 \\ &\ \ 6x_1 + \quad 2x_2 + \quad 3x_3 \qquad - x_5 = \quad 90 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Notice that if $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ is a solution to the second constraint set, then, since x_4^* and x_5^* are restricted to nonnegative values, $30x_1^* + 100x_2^* + 85x_3^* = 2500 - x_4^* \leq 2500$ and $6x_1^* + 2x_2^* + 3x_3^* = 90 + x_5* \geq 90$. Therefore (x_1^*, x_2^*, x_3^*) is a solution to the first constraint set. Similarly, if (x_1^*, x_2^*, x_3^*) is a solution to the first constraint set, there exist x_4^* and x_5^* [let $x_4^* = 2500 - (30x_1^* + 100x_2^* + 85x_3^*)$ and $x_5^* = 6x_1^* + 2x_2^* + 3x_3^* - 90$] that are nonnegative and such that $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$ is a solution to the second constraint set. Thus solutions of the two constraint sets correspond, with corresponding solutions having the same first three coordinates. At the same time, the form to be minimized, $3x_1 + 2x_2 + 4x_3$, depends only on the first three coordinates. Hence the minimal value of the linear function for both problems will be the same, and points where this minimum is achieved for one problem will correspond to points with this same property for the other problem.

Clearly, this technique generalizes. Given any problem with a system of constraints containing inequalities, by adding additional nonnegative variables, an equivalent problem can be formulated with a constraint system consisting only of equalities. The number of variables added would equal the number of inequalities in the system of constraints. The variables added are called *slack variables*. In fact, they usually can be interpreted as measuring the slack or surplus of the items or requirements of the problem. For example, in the preceding diet problem, suppose the first restriction comes from consideration of the calorie intake and the second from the protein intake. Then, for a fixed diet, the slack variable x_4 measures the number of calories below the maximum calorie requirement, and x_5 measures the number of units of protein above the minimum protein requirement for that diet.

Second, suppose a linear programming problem seeks to maximize the linear function $c_1x_1 + c_2x_2 + \cdots + c_nx_n$. But the problem of maximizing this function is equivalent to the problem of minimizing its negative: $-c_1x_1 - c_2x_2 - \cdots - c_nx_n$.

Thus a maximization problem can be easily formulated as a minimization problem by multiplying the function to be optimized by (-1).

The last restriction on the standard form of the linear programming problem is that all the variables be nonnegative. For most problems this restriction comes naturally from the physical interpretation of the variables. In all the examples we have considered, the variables could assume only nonnegative values. However, for some complicated production systems involving various processes and options, it could be that some commodity that is input for some process is output for another, and it is not clear whether this commodity will be input or output in the optimal operation of the system. Thus we may wish to formulate the problem with a variable not restricted in sign. (Problems with unrestricted variables also appear when discussing duality, as we will see in Chapter 4.)

Suppose that x_1 is a variable unrestricted in sign for a linear optimization problem. However, any number can be written as the difference of two (not unique) nonnegative numbers. (For example, $7 = 7-0 = 8-1, -7 = 0-7 = 1-8$.) Hence we can introduce into the problem two nonnegative variables, say x_1' and x_1'', and replace x_1 everywhere in the problem with the difference $x_1' - x_1''$. This will give an equivalent problem with the unrestricted variable replaced by two nonnegative variables.

As a result of these methods, for any linear programming problem, an equivalent problem can be constructed that is in standard form.

Example 3.1.1. The problem of maximizing $3x_1 - 2x_2 - x_3 + x_4 - 87$ subject to

$$\begin{array}{rcrcrcrcl} 4x_1 & - & x_2 & & & + & x_4 & \leq & 6 \\ -7x_1 & + & 8x_2 & + & x_3 & & & \geq & 7 \\ x_1 & + & x_2 & & & + & 4x_4 & = & 12 \\ \multicolumn{9}{l}{x_1, x_2, x_3 \geq 0,\ x_4 \text{ unrestricted}} \end{array}$$

is equivalent to

$$\begin{array}{l} \text{Minimize } -3x_1 + 2x_2 + x_3 - (x_4' - x_4'') + 87 \\ \text{subject to} \\ \begin{array}{rcrcrcrcrcrcrcl} 4x_1 & - & x_2 & & & + & x_4' & - & x_4'' & + & x_5 & & & = & 6 \\ -7x_1 & + & 8x_2 & + & x_3 & & & & & & & - & x_6 & = & 7 \\ x_1 & + & x_2 & & & + & 4x_4' & - & 4x_4'' & & & & & = & 12 \end{array} \\ x_1, x_2, x_3, x_4', x_4'', x_5, x_6 \geq 0 \end{array}$$

In a linear programming problem, the function to be optimized is called the *objective function*. Any point $(x_1, x_2, \ldots, x_n)$ with nonnegative coordinates that satisfies the system of constraints is called a *feasible solution* to the problem. For a particular problem, a feasible solution can be interpreted as a way of operating the system under study so that all of the requirements are fulfilled, that is, as a feasible way of operation.

Thus our basic problem is to determine, from among the set of all feasible solutions, a point that minimizes the objective function. Moreover, to be able to handle involved real-life problems, we need a solution algorithm easily programmed for computer use. Existence theorems derived from, say, the theory of continuous functions on compact sets or the theory of linear functions on convex sets, although mathematically quite attractive, do not provide an efficient means for actually finding a desired solution.

The method that will be developed in this chapter for solving the basic linear programming problem is called the *simplex method.* It is credited to George Dantzig [4], and this method and its various modifications remain among the primary means used today to solve linear optimization problems. One additional feature of this method that is useful for practical application and also very attractive mathematically is that the method can handle exceptional cases. For example, the method can determine if a problem has, in fact, any feasible solutions and, if so, whether the objective function actually assumes a minimum value.

The basic step in the simplex method is derived from the pivot operation used to solve linear equations. In the next section we pause briefly from our consideration of the standard linear programming problem to consider linear equations.

Problem Set 3.1

1. (a) In Example 3.1.1, $x_1 = 4, x_2 = 12, x_3 = 0, x_4' = 21, x_4'' = 22, x_5 = 3, x_6 = 61$ is a solution to the second constraint set. Find the corresponding solution to the first constraint set.

   (b) Conversely, $x_1 = 1, x_2 = 3, x_3 = 5, x_4 = 2$ is a solution to the first constraint set. Find a corresponding solution to the second. In this case, is your answer unique?

2. Explain why the following constraint sets are not equivalent.

Set A	Set B
$x_1 + x_2 \leq 6$	$x_1 + x_2 + x_3 = 6$
$x_1 + 2x_2 \leq 10$	$x_1 + 2x_2 + x_3 = 10$
$x_1, x_2 \geq 0$	$x_1, x_2, x_3 \geq 0$

   Hint. $x_1 = 3$ and $x_2 = 3$ satisfy the inequalities of Set A. Can you find an x_3 such that $(3, 3, x_3)$ satisfies the equalities of Set B?

   This shows that when introducing slack variables, the same variable cannot be used for different inequalities.

3. Put the following problems into standard form.

   (a) Maximize $3x_1 - 2x_2$
   subject to

$$\begin{aligned} 5x_1 + 2x_2 - 3x_3 + x_4 &\leq 7 \\ 3x_2 - 4x_3 \quad\quad &\leq 6 \\ x_1 \quad\quad + x_3 - x_4 &\geq 11 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(b) Minimize $x_2 + x_3 + x_4$
subject to

$$\begin{array}{rcrcrcrl} x_1 & + & x_2 & & & & & \geq 6 \\ & & x_2 & + & x_3 & - & x_4 & \leq 1 \\ 5x_1 & - & 6x_2 & + & 7x_3 & - & 8x_4 & \geq 2 \end{array}$$

$x_1 \geq 0, x_2 \leq 0, x_3, x_4$ unrestricted

(c) Minimize $x_1 + x_3 - x_4 + 48$
subject to

$$\begin{array}{rcrcrcrcr} -3x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & -50 \\ x_1 & - & x_2 & & & + & x_4 & \leq & 100 \\ & & 2x_2 & - & x_3 & - & x_4 & \geq & -150 \end{array}$$

$x_1, x_2, x_3, x_4 \geq 0$

(d) Maximize $6x_1 - 2x_2 + 9x_3 + 300$
subject to

$$\begin{array}{rcrcrcr} 2x_1 & - & 6x_2 & - & x_3 & \leq & 100 \\ x_1 & + & x_2 & + & 9x_3 & \leq & 200 \end{array}$$

$0 \leq x_1 \leq 50, x_2 \geq -60, x_3 \geq 5$

(e) Minimize $6x_1 + x_2$
subject to

$$\begin{array}{rcrcr} -5x_1 & + & 8x_2 & \leq & 80 \\ x_1 & + & 2x_2 & \geq & 4 \end{array}$$

$x_1 \leq 10, x_2 \geq 0$

(f) Maximize $x_1 + 2x_2 + 4x_3$
subject to

$|4x_1 + 3x_2 - 7x_3| \leq x_1 + x_2 + x_3$

$x_1, x_2, x_3 \geq 0$

(g) Maximize $x_1 + 6x_2 + 12x_3$
subject to

$-x_1 - x_2 + x_4 \geq$ maximum of $7x_1 + 2x_2$ and $5x_2 + x_3 + x_4$

$x_1, x_2, x_3, x_4 \geq 0$

(h) $-x_1 - x_2 + 2x_3 + x_5$
subject to

$$\begin{array}{rcrcrcrcr} x_1 & + & 7x_2 & + & 16x_3 & \leq & 4x_4 & + & x_5 \\ & & x_3 & + & 12x_4 & \geq & x_1 & + & 6x_2 \\ & & & & 9x_5 & \leq & x_2 & + & 3x_4 \end{array}$$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

4. Determine all feasible solutions to the linear programming problem of Problem 3(a) for which

(a) $x_1 = x_2 = x_4 = 0$
(b) $x_2 = 0, x_3 = 6$
(c) $x_3 = 0$

5. Many times the amount of slack or surplus of a commodity enters into the initial formulation of the problem; it is a factor in the function to be optimized. For example, in a production problem, there could be a cost associated with the storage of the surplus production of a commodity. For another example, formulate the mathematical model for the following.

   Two warehouses supply two retail outlets with 100-lb bags of lime. Warehouse A has 1000 bags, and Warehouse B has 2000 bags. Both outlets need 1200 bags. The transportation costs in cents per bag are given in the following table.

From	*Outlet 1*	*Outlet 2*
Warehouse A	5	4
Warehouse B	12	9

   However, there is a storage charge of 2 cents/bag for all bags left at Warehouse A and 8 cents/bag for those left at Warehouse B. Determine a shipping schedule that minimizes the total cost.

6. In the text it was suggested that when putting a linear programming problem with unrestricted variables into standard form, each unrestricted variable is to be replaced by a pair of nonnegative variables. Actually, this method is inefficient if the problem has more than one unrestricted variable; we need introduce only one additional variable to handle all the unrestricted variables. For example, if a problem has unrestricted variables x_1 and x_2, show that replacing x_1 with $x_1' - x_0$ and x_2 with $x_2' - x_0$ where x_1', x_2' and x_0 are new nonnegative variables leads to an equivalent problem.

7. Show that the following problems are equivalent.

Problem A: Minimize $x_1 + 2x_2 - 3x_3 + 4x_4$
subject to

$$\begin{aligned} 3x_1 - 2x_2 + 5x_3 - 6x_4 &= 20 \\ x_1 + 7x_2 - 6x_3 + 9x_4 &= 30 \end{aligned}$$

$x_1 \geq 0, x_2, x_3, x_4$ unrestricted

Problem B: Minimize $x_1 + 2x_2' - 3x_3' + 4x_4' - 3x_0$
subject to

$$\begin{aligned} 3x_1 - 2x_2' + 5x_3' - 6x_4' + 3x_0 &= 20 \\ x_1 + 7x_2' - 6x_3' + 9x_4' - 10x_0 &= 30 \end{aligned}$$

$x_1, x_2', x_3', x_4', x_0 \geq 0$

8. Using the technique suggested in Problem 6, determine a linear programming problem in standard form with only eight variables and equivalent to the linear programming problem of Problem 3(b).

3.2 LINEAR EQUATIONS AND BASIC FEASIBLE SOLUTIONS

The pivot operation used in solving linear equations consists of replacing a system of equations with an equivalent system in which a selected variable is eliminated from all but one of the equations. The operation revolves around what is called the *pivot term.* The pivot term can be the term in any one of the equations that contains the selected variable with a nonzero coefficient. In the first step of the pivot operation, the equation containing the pivot term is divided by the coefficient in that term, thus producing an equation in which the selected variable has coefficient 1. Multiples of this equation are added to the remaining equations in such a way that the selected variable is eliminated from these remaining equations.

It is easy to show that the solution set of the system of equations resulting from the pivot operation is identical to the solution set of the original system, that is, that the systems are equivalent (Problem 9). In general, repeated use of this pivot operation can lead to a system of equations whose solution set is obvious.

Example 3.2.1. Solve

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 6 \\ 3x_1 + 14x_2 + 8x_3 &= 16 \\ 4x_1 + 21x_2 + 10x_3 &= 28 \end{aligned}$$

We arbitrarily select x_1 as the first variable to be eliminated from two of the equations and the $1x_1$ term of the first equation as the pivot term. Notice that we could have also selected the $3x_1$ term of the second equation or the $4x_1$ term of the third equation for the pivot term. However, the arithmetic associated with the selection of the $1x_1$ term is less involved because of the unit coefficient. The pivot operation at this term consists of dividing the first equation by 1, subtracting three times the first equation from the second, and subtracting four times the first equation from the third. The resulting equivalent system is

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 6 \\ 2x_2 + 2x_3 &= -2 \\ 5x_2 + 2x_3 &= 4 \end{aligned}$$

Continuing, we arbitrarily select x_2 as the next variable to be eliminated from two of the equations. Since we are striving to simplify the system, the next pivot term should not be the $4x_2$ term of the first equation; pivoting here would reinstate in the last two equations the x_1 variable. Pivoting at the x_2 term of either of the other two equations, however, will isolate the x_2 variable to that pivoting equation without destroying the isolated status of the x_1 variable. Using the $2x_2$ term of the second equation as the pivot term (i.e., we divide the second equation by 2, then subtract four times the result from the first equation and five times the result from the third equation), we obtain

$$\begin{aligned} x_1 \qquad - 2x_3 &= 10 \\ x_2 + x_3 &= -1 \\ - 3x_3 &= 9 \end{aligned}$$

At this stage, one might solve the third equation for x_3 and use this value and the first two equations to compute the associated values for x_1 and x_2. Actually, that operation is essentially equivalent to the pivot operation with the $-3x_3$ term of the third equation as pivot term. Pivoting at this term gives

$$\begin{aligned} x_1 \qquad\qquad &= 4 \\ x_2 \qquad &= 2 \\ x_3 &= -3 \end{aligned}$$

and this system of equations is equivalent to the original system. However, the solution set for the system obviously consists only of the point $(4, 2, -3)$, so we have proven that this point is the unique solution to the original problem.

As we have seen in this example, repeated use of the pivot operation led to a system of three equations with three unknowns in a special form, where each variable appeared in one and only one equation and in that equation had coefficient 1. This form, called the *canonical form*, is crucial to the simplex method. We now define it, along with the associated term *basic variable*.

Definition 3.2.1. A system of m equations and n unknowns, with $m \leq n$, is in *canonical form* with a distinguished set of *m basic variables* if each basic variable has coefficient 1 in one equation and 0 in the others, and each equation has exactly one basic variable with coefficient 1.

Given a linear programming problem in standard form, one way of simplifying the problem would be to replace the set of constraints with an equivalent system of equations in canonical form. Indeed, this step is necessary before the simplex algorithm can be initiated on the linear programming problem. To apply the algorithm, the system of constraints must be in canonical form and the associated basic solution must be feasible. We define the terms *basic solution* and *basic feasible solution* in the following example.

Example 3.2.2. Consider the linear programming problem in standard form of

$$\begin{aligned} &\text{Minimizing } x_1 - x_2 + 2x_3 - 5x_4 = f(x_1, x_2, x_3, x_4) \qquad (3.2.1) \\ &\text{subject to} \\ &x_1 + x_2 + 2x_3 + x_4 = 6 \\ &\qquad 3x_2 + x_3 + 8x_4 = 3 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The system of constraints consists of two equations in four unknowns. Pivoting at the $3x_2$ term of the second equation gives the equivalent system

$$\begin{aligned} x_1 \quad\; + \tfrac{5}{3}x_3 - \tfrac{5}{3}x_4 &= 5 \\ x_2 + \tfrac{1}{3}x_3 + \tfrac{8}{3}x_4 &= 1 \end{aligned} \tag{3.2.2}$$

This system is in canonical form with basic variables x_1 and x_2. One particular solution to this system of equations is obvious: set the nonbasic variables x_3 and x_4 equal to 0, and set x_1 equal to the constant term 5 and x_2 equal to the constant term 1. This solution point is called a *basic feasible solution.*

Given a system of equations in canonical form with a specified set of basic variables, the associated *basic solution* is that solution to the system with the values of the basic variables given by the constant terms in the equations and the values of the nonbasic variables equal to zero.

In a linear programming problem we are interested in solutions to the system of constraints with nonnegative coordinates. Those basic solutions with this property we call *basic feasible solutions.* These will prove to be the critical points when using the simplex method to determine the optimal value of the objective function.

The point $(5,1,0,0)$ is not the only basic feasible solution for the problem in our example. Returning to the constraints of (3.2.1), if we pivot at the $8x_4$ term of the second equation instead of the $3x_2$ term (or if we pivot in (3.2.2) at the $\frac{8}{3}x_4$ term of the second equation), we get

$$\begin{aligned} x_1 + \tfrac{5}{8}x_2 + \tfrac{15}{8}x_3 \quad\;\; &= \tfrac{45}{8} \\ \tfrac{3}{8}x_2 + \tfrac{1}{8}x_3 + x_4 &= \tfrac{3}{8} \end{aligned} \tag{3.2.3}$$

Here the constraint set is represented by a system of equations in canonical form with basic variables x_1 and x_4, and the associated basic solution $(\frac{45}{8},0,0,\frac{3}{8})$ is another basic feasible solution.

Pivoting at the $\frac{5}{8}x_2$ term of the first equation in (3.2.3) yields the equivalent system

$$\begin{aligned} \tfrac{8}{5}x_1 + x_2 + 3x_3 \quad\;\; &= 9 \\ -\tfrac{3}{5}x_1 \qquad - x_3 + x_4 &= -3 \end{aligned}$$

This system is in canonical form with basic variables x_2 and x_4, but the associated basic solution $(0,9,0,-3)$ is not feasible. The value of x_4 is negative. Obviously, randomly selecting the variables to serve as basic variables can lead to a system of equations with some negative constant terms and thus an associated basic solution that is not feasible. As we will see, the simplex method provides a systematic way to resolve the problem of starting with and maintaining feasibility.

We return now to the original linear programming problem of (3.2.1), but with the system of constraints replaced by the equivalent system of (3.2.2), a system in canonical form with a basic feasible solution. In order to apply the simplex method to the problem, one final step involving the objective function is necessary. The expression for the objective function needs to be coordinated with the canonical form of the system of the constraints. In particular, the expression for the objective function must be in terms of only the nonbasic variables. This step can be considered an extension of the pivot operation used to put the system of constraints into canonical form, and is easily accomplished here using the system of constraints. We demonstrate.

The objective function of the example is

$$f(x_1,x_2,x_3,x_4) = x_1 - x_2 + 2x_3 - 5x_4$$

and the system of constraints in canonical form with basic variables x_1 and x_2, from (3.2.2), is

$$\begin{aligned} x_1 \quad + \tfrac{5}{3}x_3 - \tfrac{5}{3}x_4 &= 5 \\ x_2 + \tfrac{1}{3}x_3 + \tfrac{8}{3}x_4 &= 1 \end{aligned}$$

From these equations, it is obvious that the value of the objective function f at any point (x_1,x_2,x_3,x_4) satisfying the constraints can be given by

$$\begin{aligned} x_1 - x_2 + 2x_3 - 5x_4 &= \left[5 - \tfrac{5}{3}x_3 + \tfrac{5}{3}x_4\right] - \left[1 - \tfrac{1}{3}x_3 - \tfrac{8}{3}x_4\right] + 2x_3 - 5x_4 \\ &= \tfrac{2}{3}x_3 - \tfrac{2}{3}x_4 + 4 \end{aligned}$$

Thus on this system of constraints, the problem of minimizing f is equivalent to the problem of minimizing the function $\frac{2}{3}x_3 - \frac{2}{3}x_4 + 4$. With this new function our goal of expressing the function to be optimized in terms of only the nonbasic variables is attained.

Through these operations we have replaced the linear programming problem of (3.2.1) with the following equivalent linear programming problem.

$$\begin{aligned} &\text{Minimize } \tfrac{2}{3}x_3 - \tfrac{2}{3}x_4 + 4 \\ &\text{subject to} \\ &x_1 \quad + \tfrac{5}{3}x_3 - \tfrac{5}{3}x_4 = 5 \\ &\quad x_2 + \tfrac{1}{3}x_3 + \tfrac{8}{3}x_4 = 1 \\ &x_1,x_2,x_3,x_4 \geq 0 \end{aligned}$$

This problem is said to be in *canonical form* with basic variables x_1 and x_2.

Definition 3.2.2. The standard linear programming problem is in *canonical form* with a distinguished set of basic variables if:

(a) The system of constraints is in canonical form with this distinguished set of basic variables.

(b) The associated basic solution is feasible.

(c) The objective function is expressed in terms of only the nonbasic variables.

If the first two conditions of this definition are satisfied for a linear programming problem, the system of constraints can be used, as in the above example, to eliminate the basic variables from the objective function. While organizing and maintaining a problem in canonical form, we will abuse the language somewhat and always speak of one fixed objective function. Certainly in the above example the function $x_1 -$

$x_2 + 2x_3 - 5x_4$ does not equal the function $\frac{2}{3}x_3 - \frac{2}{3}x_4 + 4$. However, the problems of optimizing these functions on the given constraint set are equivalent, that is, the functions have the same minimum value, and the sets of feasible solutions on which this common optimal value is attained are the same. It is this equivalency that we have in mind when we say, for example, that the objective function is now given by $\frac{2}{3}x_3 - \frac{2}{3}x_4 + 4$.

The question of feasibility of a basic solution can be stated geometrically using the column vectors associated with the coefficient matrix of the system of equations. We demonstrate.

Example 3.2.3. The system of constraints for the linear programming problem of (3.2.1) can be expressed in vector form as follows:

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Thus the system of two equations and four variables is equivalent to the problem of expressing the vector $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$. Moreover, for our purposes, we are restricted to solutions with nonnegative coordinates.

Suppose now we wish to determine geometrically if x_1 and x_2 can serve as basic variables for a basic feasible solution. If so, the nonbasic variables x_3 and x_4 will equal zero, and the resulting vector equation reduces to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad x_1, x_2 \geq 0$$

Using the notation

$$A^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

these vectors in $\mathbb{R}^2$ are sketched in Figure 3.1.

Now the set of points of the form $x_1 A^{(1)}$ for $x_1 \geq 0$ is the line ray emanating from the origin in $\mathbb{R}^2$ in the direction of $A^{(1)}$, and similarly for the points $x_2 A^{(2)}$ with $x_2 \geq 0$. The set of points of the form $x_1 A^{(1)} + x_2 A^{(2)}$, x_1 and $x_2 \geq 0$, can be determined using the usual rule for addition of vectors. This region (the *convex cone* of $A^{(1)}$ and $A^{(2)}$) is illustrated in Figure 3.2. Since b lies in this region, a solution to the system of equations with x_1 and x_2 nonnegative and x_3 and x_4 equal to 0 must exist. This solution is the point $(5, 1, 0, 0)$ found previously.

To extend these ideas, let $A^{(3)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $A^{(4)} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$. From the graph in Figure 3.3 we see that b cannot be expressed as a sum of the form $x_2 A^{(2)} + x_4 A^{(4)}$ with x_2 and $x_4 \geq 0$. Thus x_2 and x_4 cannot serve as basic variables for a basic feasible solution.

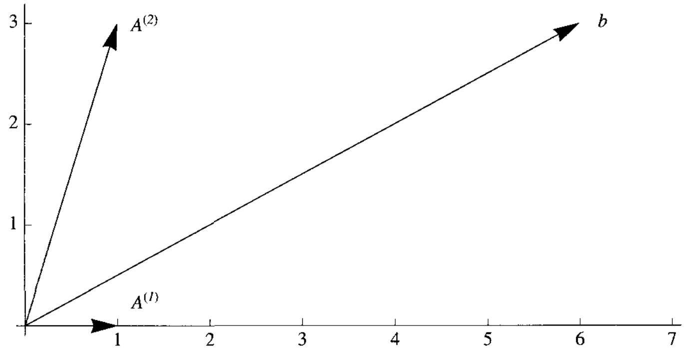


Figure 3.1

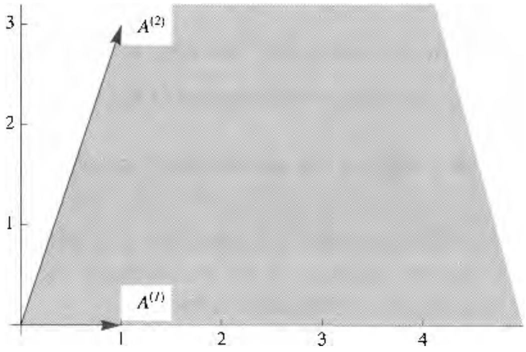


Figure 3.2

(Recall, the associated basic solution is $(0,9,0,-3)$.) Furthermore, it can be seen that any other pair of variables can serve as basic variables for a basic feasible solution. Note also that b is a multiple of $A^{(3)}$ alone. Thus in any basic feasible solution with x_3 as a basic variable, only the x_3 coordinate will be nonzero. Indeed, pivoting at the $1x_3$ term in the second equation in the constraints of (3.2.1) yields the equivalent system

$$\begin{aligned} x_1 - 5x_2 \qquad - 15x_4 &= 0 \\ 3x_2 + x_3 + \ \ 8x_4 &= 3 \end{aligned}$$

This system is in canonical form with basic variables x_1 and x_3, and the associated basic (feasible) solution is $(0,0,3,0)$, with the basic variable x_1 equal to zero. A basic solution with some basic variables equal to zero is said to be *degenerate*. As we will see later when developing the simplex method, theoretical complications arise from the possibility of degeneracy.

The reader may be somewhat puzzled by our earlier remark that, when determining the minimum of the objective function of a linear programming problem, the

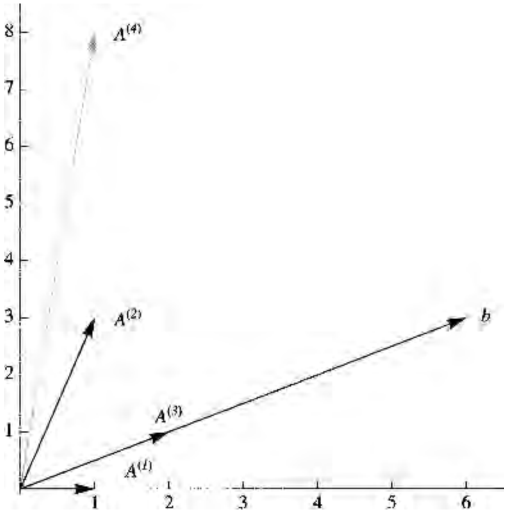


Figure 3.3

basic feasible solutions are the critical points to be considered. Why, when trying to minimize a function, should we wish to restrict our attention to only those feasible solutions of the constraint set that are basic and therefore have at least $n-m$ zero coordinates? For example, in a diet problem with five nutritional requirements and 15 foods from which to choose, is it possible to find a minimal-cost diet that uses at most only 5 of the foods? As we will show in this chapter, the answer to this question is "yes." In fact, we will show by an algebraic argument that if the objective function does have a minimum value, that value is assumed by at least one basic feasible solution.

Actually, the role played by the basic feasible solutions in the resolution of a two-variable problem is apparent from the geometry of such a problem. Consider, for example, the solution procedure used to solve the blending problem developed in Example 2.2.1 on page 10. The problem there was to determine a blend of two feeds that minimized costs and met three nutritional requirements. Letting x_1 denote the amount of Feed 1 and x_2 the amount of Feed 2 in a diet, the associated mathematical problem was to

$$\begin{aligned}
&\text{Minimize } 10x_1 + 4x_2\\
&\text{subject to}\\
&3x_1 + 2x_2 \geq 60\\
&7x_1 + 2x_2 \geq 84\\
&3x_1 + 6x_2 \geq 72\\
&x_1, x_2 \geq 0
\end{aligned}$$

Putting this into standard form gives the following:

$$\begin{aligned}
&\text{Minimize } 10x_1 + 4x_2 \\
&\text{subject to} \\
&3x_1 + 2x_2 - x_3 = 60 \\
&7x_1 + 2x_2 - x_4 = 84 \\
&3x_1 + 6x_2 - x_5 = 72 \\
&x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned} \tag{3.2.4}$$

The slack variables x_3, x_4, and x_5 measure the surplus amounts of the nutritional elements A, B, and C in a given diet. Now the geometric argument based on Figure 2.5 on page 13 showed that if the linear function had a minimal value, the function would assume that value at a corner or vertex of the region shaded in Figure 2.3. The four vertices of the shaded region in Figure 2.3 are the points $(0,42)$, $(6,21)$, $(18,3)$, and $(24,0)$. They occur on the boundaries of the regions defined by the original three inequalities, that is, when some of the inequalities are actually equalities and the corresponding slack variables therefore equal zero. In fact, the solutions to the constraint set in standard form corresponding to these four points are:

$$\begin{aligned}
(0,42) &\leftrightarrow (0,42,24,0,180) \\
(6,21) &\leftrightarrow (6,21,0,0,72) \\
(18,3) &\leftrightarrow (18,3,0,48,0) \\
(24,0) &\leftrightarrow (24,0,12,84,0)
\end{aligned}$$

Note that each of the four points in the right column has two coordinates at zero level. These four points are basic feasible solutions to the constraint set in standard form. Therefore, if the objective function is bounded below, the minimal value must occur at a basic feasible solution.

This geometrical analysis extends to the general problem, yielding another proof that for a linear programming problem, if the set of optimal solution points is not empty, the set of basic feasible solutions provides the foundation for this set. However, we do not use these ideas in the algebraic development which follows, and so we will postpone discussion of the geometry of the general problem until Section 3.9.

Problem Set 3.2

1. Solve the following using the pivot operation.

(a)
$$\begin{aligned}
3x_2 - 3x_3 &= 15 \\
x_1 + x_2 + x_3 &= 0 \\
3x_1 + 5x_2 + 3x_3 &= 4
\end{aligned}$$

(b)
$$\begin{aligned}
3x_1 + 2x_2 - 7x_3 &= 1 \\
x_1 - 5x_2 - 6x_3 &= -4
\end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 \qquad - 2x_4 &= 5 \\ - 3x_2 + x_3 + 4x_4 &= 2 \end{aligned}$$

2. A system of equations is said to be *redundant* if one of the equations in the system is a linear combination of the other equations. Show by using the pivot operation that the following system is redundant. Is this system equivalent to a system of equations in canonical form?

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ -2x_1 + x_2 + 5x_3 &= 2 \\ 3x_2 - x_3 &= 16 \end{aligned}$$

3. A system of equations is said to be *inconsistent* if the system has no solution. Show by using the pivot operation that the following systems are inconsistent. Is either of these systems equivalent to a system in canonical form?

(a) $$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_2 - 3x_3 &= 7 \\ -2x_1 + x_2 + 5x_3 &= 2 \\ 3x_2 - x_3 &= 15 \end{aligned}$$

4. (a) Solve the following system of equations by finding an equivalent system in canonical form with basic variables x_1 and x_2.

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 17 \\ x_1 \qquad - x_3 &= 4 \end{aligned}$$

(b) Is this system equivalent to a system in canonical form with basic variables x_1 and x_3?

(c) Interpret these results geometrically.

5. Suppose a system of equations contains the following terms:

$$\begin{aligned} &ax_1 + bx_2 \\ &cx_1 + dx_2 \end{aligned}$$

where a, b, c, and d are constants, $a \neq 0$.

The system is then replaced with an equivalent system by pivoting at the ax_1 term. Show that these four terms become

$$\begin{aligned} &x_1 + \frac{b}{a}x_2 \\ &0x_1 + \left(d - \frac{bc}{a}\right)x_2 \end{aligned}$$

The expression $d - bc/a$ provides a way of remembering the effect of the pivot operation on any term not in the row or column of the pivot term.

6. For the linear programming problem of

$$\begin{aligned}&\text{Minimizing } 5x_1 + 2x_2 + 3x_3 + x_4\\&\text{subject to}\\&\quad x_1 + x_2 - 2x_3 + 3x_4 = 2\\&-2x_1 \qquad\quad + x_3 \qquad\quad = 2\\&x_1, x_2, x_3, x_4 \geq 0\end{aligned}$$

(a) Show geometrically that there can be only two basic feasible solutions to the problem.
(b) Compute these two basic feasible solutions.
(c) Show that the objective function is bounded below.
(d) Assume that the minimal value of the objective function is attained at a basic feasible solution and determine this minimal value.

7. Following the outline in Problem 6, complete the problem of Example 3.2.3.

8. (a) Put the constraint set from the standard form of the blending problem considered in this section (the problem of (3.2.4)) into canonical form with basic variables x_1, x_2, and x_5. The associated basic feasible solution is $(6,21,0,0,72)$.
(b) The objective function for this problem is $10x_1 + 4x_2$. By eliminating the x_1 and x_2 variables by using the equations found in part (a), this function can be expressed in terms of only x_3 and x_4. Verify that the form reduces to $144 + x_3 + x_4$.
(c) Since we are considering only feasible solutions to the constraint set, using part(b), give another proof that the minimal value of the objective function is 144.

9. Prove that the system of equations resulting from a given system by applying the pivot operation is equivalent to (has the same solution set as) the original system.

10. Prove that although there may be different ways of driving a system of equations into canonical form with a specified set of basic variables, there is a unique basic solution associated with this specified set of basic variables.

11. True or false: A system of equations is equivalent to a system of equations in canonical form if and only if the original system has at least one solution.

12. Construct a linear programming problem with four variables and three equations for which there exist degenerate feasible solutions with exactly two nonzero coordinates.

3.3 INTRODUCTION TO THE SIMPLEX METHOD

In this section the simplex method for solving linear programming problems will be introduced. The basic ideas behind the technique will be demonstrated by means of a specific example. The goal of this section is to develop motivation and understanding; the theorems related to the simplex method will be proven in subsequent sections of this chapter.

Let us consider the following problem in standard form:

$$\begin{array}{l} \text{Minimize } -4x_1+x_2+x_3+7x_4+3x_5=z \\ \text{subject to} \\ \begin{array}{rrrrrcl} -6x_1 & & +\,x_3 & -\,2x_4 & +\,2x_5 & = & 6 \\ 3x_1 & +\,x_2 & -\,x_3 & +\,8x_4 & +\,x_5 & = & 9 \end{array} \\ x_1,x_2,x_3,x_4,x_5 \geq 0 \end{array} \tag{3.3.1}$$

The simplex method can begin only with the problem in canonical form. To put the problem into canonical form, we could first arbitrarily select two variables to be basic variables and then, by pivoting, attempt to put the system of constraints into canonical form with these variables as basic variables, with the hope that the associated basic solution would be feasible. Or, because here we have a problem with only two constraints, we could determine, using elementary vector geometry, a pair of variables that would serve as basic variables for a feasible solution.

In general, however, finding an initial basic feasible solution to a problem can be a major difficulty. This problem will be solved in Section 3.6. For now, assume that we know that for the problem at hand, the variables x_2 and x_3 can serve as basic variables for a feasible solution. Pivoting at the $1x_3$ term of the first equation will put the system of constraints into canonical form. This gives

$$\begin{array}{rrrrrcl} -6x_1 & & +\,x_3 & -\,2x_4 & +\,2x_5 & = & 6 \\ -3x_1 & +\,x_2 & & +\,6x_4 & +\,3x_5 & = & 15 \end{array} \tag{3.3.2}$$

The associated basic solution, $(0,15,6,0,0)$, is feasible, as promised. Now these two equations can be used to eliminate the basic variables x_2 and x_3 from the expression for the objective function z, given by

$$-4x_1+x_2+x_3+7x_4+3x_5=z \tag{3.3.3}$$

In fact, simply subtracting the two equations in (3.3.2) from the equation in (3.3.3) gives

$$5x_1+0x_2+0x_3+3x_4-2x_5=z-21$$

Hence the objective function can be given by the form

$$5x_1+3x_4-2x_5+21=z$$

Thus the problem in canonical form with basic variables x_2 and x_3 is to

$$\begin{array}{l} \text{Minimize } z \text{ with} \\ \begin{array}{rrrrrcl} -6x_1 & & +\,x_3 & -\,2x_4 & +\,2x_5 & = & 6 \\ -3x_1 & +\,x_2 & & +\,6x_4 & +\,3x_5 & = & 15 \\ 5x_1 & & & +\,3x_4 & -\,2x_5 & = & -21+z \end{array} \end{array} \tag{3.3.4}$$

The objective function has the value 21 at the associated basic feasible solution $(0,15,6,0,0)$. Now the key idea behind the simplex method is to move to another basic feasible solution that gives a smaller value for z by replacing exactly one basic variable from the present set. As we will see, the mechanics for this replacement will be provided by the pivot operation. However, what variable from the set of nonbasic variables x_1, x_4, and x_5 to insert into the basis, and what basic variable, x_2 or x_3, to replace in order to reduce the value of z are not obvious.

These questions are answered first by considering the objective function $z = 5x_1 + 3x_4 - 2x_5 + 21$. In this expression for z, the x_5 variable has a negative coefficient. Thus a feasible solution to the constraint set with x_1 and x_4 still equal to zero, but with x_5 greater than zero, will give a smaller value for z. This suggests that we move x_5 into the set of basic variables and attempt to make x_5 as large as possible.

But what basic variable, x_1 or x_3, should we replace? To answer this question, consider the constraint set with the conditions imposed by this situation, that the nonbasic variables x_1 and x_4 equal zero. From (3.3.4) we have

$$\begin{aligned} x_3 + 2x_5 &= 6 \\ x_2 + 3x_5 &= 15 \end{aligned}$$

Solving for x_3 and x_1 gives

$$\begin{aligned} x_3 &= 6 - 2x_5 \\ x_2 &= 15 - 3x_5 \end{aligned} \tag{3.3.5}$$

Clearly, x_5 cannot be arbitrarily large. To have a solution to the constraint set with $x_1 = x_4 = 0$, x_2 and x_3 must satisfy these equations and would possibly become negative. In fact, since x_2 and x_3 must be nonnegative, x_5 is restricted by the inequalities

$$0 \le 6 - 2x_5 \quad \text{and} \quad 0 \le 15 - 3x_5$$

that is, $x_5 \le 3 = \frac{6}{2}$ and $x_5 \le 5 = \frac{15}{3}$. Since x_5 must satisfy both these inequalities, the maximum possible value for x_5 is 3. Letting $x_5 = 3$ and using (3.3.5) to calculate x_3 and x_2, we have the feasible solution $x_1 = x_4 = 0$, $x_5 = 3$, $x_3 = 0$, and $x_2 = 6$. The value of z at this point is 15, six less than the value at the first basic feasible solution. At the point $(0,6,0,0,3)$, $x_2 = 6$ and $x_3 = 0$. Thus x_3, being at zero level, is the variable that should be replaced in the basis, giving x_2 and x_5 as the basic variables for this second solution point. (Note also that at $(0,6,0,0,3)$, x_2 and x_5 are the two variables assuming positive values.)

In fact, by letting x_5 equal the minimum of 3 and 5, we are guaranteed that x_3 will assume the value 0, because the minimum value 3 is the bound coming from the x_3 equation in (3.3.5). To determine the variable to extract from the basis, then, we need only determine the basic variable of that equation in the modified constraint set (3.3.5) that leads to the minimal bound. And each of these bounds of $3 = \frac{6}{2}$ and $5 = \frac{15}{3}$ is the ratio of the constant term in the equation to the coefficient of the x_5 variable. This suggests a simple procedure for determining the variable to extract from the basis, a procedure that will be spelled out in detail in the next section.

The simplex method is the continuation of this process. To proceed, however, the problem must be in canonical form with basic variables x_2 and x_5. To do this, we use the pivot operation. With the system of constraints expressed as in (3.3.4), the first equation contains the basic variable x_3, which is to be replaced with the variable x_5. Hence pivoting at the $2x_5$ term of this equation will put the system of constraints into canonical form with basic variables x_2 and x_5. Moreover, the effect of this pivot operation on the third equation in (3.3.4) would be to eliminate the variable x_5 from that equation also. Then the objective function z would be expressed in terms of only the variables x_1, x_3, and x_4. Thus the effect of the pivot operation at the $2x_5$ term of the first equation in (3.3.4) applied to all three equations would be to transform the entire problem into the desired canonical form. Pivoting here gives

$$\begin{aligned} -3x_1 \quad + \tfrac{1}{2}x_3 - x_4 + x_5 &= 3 \\ 6x_1 + x_2 - \tfrac{3}{2}x_3 + 9x_4 \quad &= 6 \\ -x_1 \quad + x_3 + x_4 \quad &= -15+z \end{aligned} \tag{3.3.6}$$

Now we proceed exactly as before. The variable x_1 has a negative coefficient in the expression for the objective function and so should be inserted into the basis. Letting $x_3 = x_4 = 0$, the constraint set of (3.3.6) becomes

$$\begin{array}{c} -3x_1 + x_5 = 3 \\ 6x_1 + x_2 = 6 \end{array} \quad \text{or} \quad \begin{array}{c} x_5 = 3 + 3x_1 \\ x_2 = 6 - 6x_1 \end{array} \tag{3.3.7}$$

Since x_2 and x_5 must be nonnegative, we have

$$\begin{array}{c} 0 \le 3 + 3x_1 \\ 0 \le 6 - 6x_1 \end{array} \quad \text{or} \quad \begin{array}{c} -1 \le x_1 \\ x_1 \le 1 \end{array}$$

The first inequality places no upper bound on x_1, so the upper limit for x_1 is determined solely by the second inequality, the inequality resulting from the x_2 equation in (3.3.7). Thus x_1 should replace x_2 in the basis. Letting $x_1 = 1$ gives the basic feasible solution $(1,0,0,0,6)$, and the value of the objective function at this point is 14.

One lingering question that we have so far avoided is the following: When do we know that the minimal value of the objective function has been achieved and the process can terminate? Our example will now provide the answer to this question.

We have seen that a reduced value for z can be determined by using x_1 and x_5 as basic variables instead of x_2 and x_5. Accordingly, we put the system into canonical form with these as basic variables by pivoting at the $6x_1$ term of the second equation in (3.3.6). This gives

$$\begin{aligned} \tfrac{1}{2}x_2 - \tfrac{1}{4}x_3 + \tfrac{7}{2}x_4 + x_5 &= 6 \\ x_1 + \tfrac{1}{6}x_2 - \tfrac{1}{4}x_3 + \tfrac{3}{2}x_4 \quad &= 1 \\ \tfrac{1}{6}x_2 + \tfrac{3}{4}x_3 + \tfrac{5}{2}x_4 \quad &= -14+z \end{aligned}$$

The objective function is given by $z = \frac{1}{6}x_2 + \frac{3}{4}x_3 + \frac{5}{2}x_4 + 14$. In contrast to the two previous situations, here the coefficients of the nonbasic variables are all positive. This means in fact that the value of the objective function at any feasible solution to the constraint set must be at least 14, since all the coordinates of a feasible solution are nonnegative. Thus our process is terminated. The minimal value of the objective function can be no less than 14, and this value is attained at the point $(1,0,0,0,6)$.

To summarize, the simplex method begins with the problem in canonical form. We move from one basic feasible solution to another by replacing exactly one basic variable at each step, with the new basic feasible solution providing a reduced value of the objective function (except possibly when there is degeneracy, a complication to be discussed later). Consideration of the coefficients of the objective function tells us if the minimal value has been achieved and, if not, what variable to insert into the basis. Consideration of the modified constraint set tells us what variable to extract from the basis. And one simple pivot operation at each step keeps the entire system in proper form.

In the next section, we will make precise the simplex method for the general problem and will consider the case where the objective function is not bounded below. (See also Problem 3.) In Section 3.6 a method based on the simplex method for determining an initial basic feasible solution will be discussed.

Problem Set 3.3

1. Consider the system of equations

$$\begin{aligned} x_1 \qquad\qquad + 2x_4 &= 8 \\ x_2 \qquad + 3x_4 &= 6 \\ x_3 + 6x_4 &= 18 \end{aligned} \tag{3.3.8}$$

   The system is in canonical form with basic variables x_1, x_2, and x_3, and the associated basic solution is feasible.

   (a) Express the set of solutions to the system in terms of x_4, that is, solve for x_1, x_2, and x_3 in terms of x_4.
   (b) Determine the set of values for the parameter x_4 for which the corresponding solutions to the system are feasible.
   (c) Let x_4 be the largest value in this set. What variable assumes the value zero?
   (d) Suppose we wish to express the system in canonical form with x_4 in the basis, and such that the associated basic solution is feasible. From (c), what variable should be extracted from the basis and become the nonbasic variable? Thus, at what term in (3.3.8) should we pivot?
   (e) Show that pivoting here has the desired effect.
   (f) For each equation in (3.3.8), compute the ratio of the constant term to the coefficient of x_4. Relate these values to the choice of pivoting term in (d).

2. Consider the problem of

$$\begin{aligned} &\text{Minimizing } x_1 + x_2 + 4x_3 + 7x_4 \\ &\text{subject to} \\ &x_1 + x_2 + 5x_3 + 2x_4 = 8 \\ &2x_1 + x_2 + 8x_3 \qquad = 14 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) The variables x_1 and x_2 can serve as basic variables for a basic feasible solution. Show that the problem expressed with these as basic variables is

$$\begin{aligned} x_1 \quad + 3x_3 - 2x_4 &= 6 \\ x_2 + 2x_3 + 4x_4 &= 2 \\ - x_3 + 5x_4 &= -8+z \end{aligned}$$

(b) Entering x_3 into the basis will reduce the value of z. Why? Show that the variable to be replaced is x_2.

(c) Perform the pivot operation. Show that the minimal value of the objective function is 7 and is achieved at $(3,0,1,0)$.

3. Use the simplex method to do the following problem. The problem is stated in canonical form with basic variables x_2 and x_3. Notice that at the first step in the simplex method, either x_1 or x_4 can enter the basis.

$$\begin{aligned} &\text{Minimize } -x_1 - 2x_4 + x_5 \\ &\text{subject to} \\ &x_1 \qquad + x_3 + 6x_4 + 3x_5 = 2 \\ &-3x_1 + x_2 \qquad + 3x_4 + \ x_5 = 3 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

4. In the following problem, the objective function does not have a minimum. However, the problem is stated in canonical form with basic variables x_1 and x_2, and the simplex method can be initiated.

$$\begin{aligned} &\text{Minimize } 4x_3 - 6x_4 \\ &\text{subject to} \\ &\quad x_2 - 6x_3 + 2x_4 = 6 \\ &x_1 \quad + 2x_3 - \ x_4 = 5 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(a) What occurs after the first pivot operation that makes this problem different from our other examples?

(b) Can you prove, using the resulting equations, that the objective function is in fact not bounded below?

3.4 THEORY OF THE SIMPLEX METHOD

In this section we develop the simplex method for a general linear programming problem. To initiate the algorithm, the problem must be in canonical form. In Section 3.1 we showed that any linear programming problem is equivalent to a problem in standard form, and in Section 3.6 we will show how to drive a problem in standard form into canonical form. In fact, the technique developed in Section 3.6 will make use of the ideas developed in this section. Thus, for the time being, we assume that our general problem is in canonical form.

Suppose the problem has m constraints and n variables, with the first m variables as basic variables. The problem is then:

$$\begin{array}{llcll}
\text{Minimize } z \text{ where} & & & & (3.4.1)\\
x_1 + \quad \dots + a_{1,m+1}x_{m+1} & + \cdots + & a_{1n}x_n & = b_1 & \\
\quad + x_2 + \dots + a_{2,m+1}x_{m+1} & + \cdots + & a_{2n}x_n & = b_2 & \\
\qquad \ddots & & & & \\
\qquad\quad x_m + a_{m,m+1}x_{m+1} & + \cdots + & a_{mn}x_n & = b_m & \\
\qquad\qquad c_{m+1}x_{m+1} & + \cdots + & c_nx_n & = z_0 + z & \\
x_1, x_2, \dots, x_n \geq 0 & & & &
\end{array}$$

a_{ij}, b_i, c_j, and z_0 are constants and, since the associated basic solution is feasible, $b_i \geq 0, i = 1, ..., m$.

Example 3.4.1. We wish to minimize z with

$$\begin{array}{rcl}
x_1 \quad + 2x_3 - \ x_4 & = & 10\\
x_2 - \ x_3 - 5x_4 & = & 20\\
2x_3 + 3x_4 & = & 60 + z\\
x_1, x_2, x_3, x_4 \geq 0 & &
\end{array}$$

Here we have a problem with $m = 2$ constraints, $n = 4$ variables, and in canonical form. The associated basic feasible solution is $(10, 20, 0, 0)$, and the value of the objective function z at this point is -60. Note that in this particular problem the coefficients $c_3 = 2$ and $c_4 = 3$ are nonnegative. Since x_3 and x_4 are restricted to be nonnegative, the smallest value $z = 2x_3 + 3x_4 - 60$ can possibly attain is -60, the value of the objective function at the $(10, 20, 0, 0)$ solution point. This suggests our first theorem.

Theorem 3.4.1 (optimality criterion). *For the linear programming problem of (3.4.1), if $c_j \geq 0$, $j = m+1, \dots, n$, then the minimal value of the objective function is $-z_0$ and is attained at the point $(b_1, b_2, \dots, b_m, 0, \dots, 0)$.*

Proof. For any point satisfying the set of constraints, the value of the objective function is given by $z = c_{m+1}x_{m+1} + \cdots + c_nx_n - z_0$. Since any feasible solution to the constraints has nonnegative coordinates, the smallest possible value for the sum $c_{m+1}x_{m+1} + \cdots + c_nx_n$ is zero. Thus the minimal possible value for z is $-z_0$, and this value is assumed at the point $(b_1, b_2, \dots, b_m, 0, \dots, 0)$. □

Hence the problem is resolved if all the c_j's are nonnegative. Assume now that at least one c_j, say c_s, is negative. Then we attempt to enter the variable x_s into the basis. In order to determine what basic variable to replace, we consider the constraint set with all the nonbasic variables except x_s equal to zero. This gives

$$\begin{array}{lcl} x_1 + a_{1s}x_s = b_1 & & x_1 = b_1 - a_{1s}x_s \\ x_2 + a_{2s}x_s = b_2 & & x_2 = b_2 - a_{2s}x_s \\ \vdots & \text{or} & \vdots \\ x_m + a_{ms}x_s = b_m & & x_m = b_m - a_{ms}x_s \end{array} \tag{3.4.2}$$

Example 3.4.2. Here we wish to minimize z with

$$\begin{array}{rcrcrcl} x_1 & & +\ 2x_3 & - & x_4 & = & 10 \\ & x_2 & -\ \ x_3 & - & 5x_4 & = & 20 \\ & & 2x_3 & - & 3x_4 & = & 60+z \\ \multicolumn{7}{l}{x_1,x_2,x_3,x_4 \geq 0} \end{array}$$

Except for a change in sign in c_4, this is exactly the problem of Example 3.4.1. As before, $(10,20,0,0)$ is a feasible solution, and the value of the objective function $z = 2x_3 - 3x_4 - 60$ at this point is -60. However, here we could reduce the value of z if we could find feasible solutions to the constraint set with x_4 positive and x_3 equal to zero, since $c_4 = -3$ is negative. Setting $x_3 = 0$, the constraints reduce to

$$\begin{array}{lcl} x_1 - \ \ x_4 = 10 & \text{or} & x_1 = 10 + \ \ x_4 \\ x_2 - 5x_4 = 20 & & x_2 = 20 + 5x_4 \end{array}$$

Note that if we fix x_4 at any positive number and then use these two equations to solve for x_1 and x_2, the resulting values will be positive. Thus all points in the set

$$\{(x_1,x_2,0,x_4) : x_4 \geq 0, x_1 = 10 + x_4, x_2 = 20 + 5x_4\}$$

are feasible solutions to the system of constraints. But the function $z = 2x_3 - 3x_4 - 60$ is unbounded below on this set. This suggests our next theorem.

Theorem 3.4.2. *For the linear programming problem of (3.4.1), if there is an index s, $m+1 \leq s \leq n$, such that $c_s < 0$ and $a_{is} \leq 0$ for all $i = 1,2,\ldots,m$, then the objective function is not bounded below.*

Proof. Assume there is an index s satisfying the conditions of the theorem. Since the coefficients a_{is} are all nonpositive, the m equations of (3.4.2) can be used to find a set S of feasible solutions to the constraints with x_s assuming arbitrarily large values, the original basic variables x_1 to x_m positive values, and the remaining variables value zero. But the objective function is given by the form

$$z = c_m x_{m+1} + \cdots + c_s x_s + \cdots + c_n x_n - z_0,$$

and on S, this reduces to $z = c_s x_s - z_0$. Since $c_s < 0$, z is unbounded below on S. $\square$

Assume now that $c_s < 0$ and that at least one $a_{is} > 0$. Then the argument above breaks down, because if $a_{is} > 0$, the equation $x_i = b_i - a_{is}x_x$ places a limit on how large x_s can become. In fact, for x_i to remain nonnegative, we must have $0 \leq b_i - a_{is}x_s$, that is, $x_s \leq b_i/a_{is}$ for $a_{is} > 0$. Thus our goal now is simply to replace in the basis one of the basic variables $x_1, \ldots, x_m$ with the variable x_s. Because of the term $c_s x_s$ in the expression for the objective function, the value of z at this new basic feasible solution hopefully will be reduced. Our one demand on this new basis is that the associated basic solution be feasible. Hence the equations of (3.4.2) for which $a_{is} > 0$ restrict our choice of the variable to extract from the basis. Since we must have $x_s \leq \frac{b_i}{a_{is}}$ for all i with $a_{is} > 0$, the largest possible value for x_s is

$$\operatorname{Min}\left\{\frac{b_i}{a_{is}} : a_{is} > 0\right\}.$$

Suppose this minimum value is attained when $i = r$. Then letting $x_s = \frac{b_r}{a_{rs}}$ will give $x_i \geq 0$ for $i = 1, \ldots, m$ and, in particular, $x_r = b_r - \frac{b_r}{a_{rs}} = 0$. Since x_r takes on the value zero here, it appears that x_r is the variable to be replaced in the basis. And since in (3.4.1) the rth equation of the constraints isolates x_r, the problem can be put into canonical form with basic variables $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_m, x_s$ by a single pivot operation at the $a_{rs} x_s$ term of the rth equation. Before formally stating and proving these results, we give an example.

Example 3.4.3. Minimize z with

$$\begin{aligned}
x_1 \qquad\qquad + 2x_4 - x_5 &= 10\\
x_2 \qquad - x_4 - 5x_5 &= 20\\
x_3 + 6x_4 - 12x_5 &= 18\\
- 2x_4 + 3x_5 &= 60 + z\\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}$$

The problem is in canonical form with basic variables x_1, x_2, and x_3. The associated basic feasible solution is $(10, 20, 18, 0, 0)$, and the value of the objective function at this point is -60. However, $c_4 = -2$ is negative, and so we attempt to reduce the value of z by inserting x_4 into the basis. Letting $x_5 = 0$, the constraints reduce to

$$\begin{aligned}
x_1 + 2x_4 &= 10 & & & x_1 &= 10 - 2x_4\\
x_2 - x_4 &= 20 & &\text{or} & x_2 &= 20 + x_4\\
x_3 + 6x_4 &= 18 & & & x_3 &= 18 - 6x_4
\end{aligned}$$

The second equation places no restriction on x_4. However, the first requires that $x_4 \leq \frac{10}{2} = 5$ and the third that $x_4 \leq \frac{18}{6} = 3$. The largest possible value for x_4 with $x_5 = 0$ is the minimum of 3 and 5, that is, 3. Letting $x_4 = 3$ gives $x_3 = 0$. Thus x_4 should replace x_3 in the basis and, since the third equation of the constraints isolates x_3, pivoting at the $6x_4$ term of this equation should keep the problem in canonical form, but with basic variables x_1, x_2, and x_4. In fact, pivoting here yields the following equivalent problem:

Minimize z with

$$\begin{array}{rcrcrcrcl} x_1 & & -\tfrac{1}{3}x_3 & & & + & 3x_5 & = & 4 \\ & x_2 & +\tfrac{1}{6}x_3 & & & - & 7x_5 & = & 23 \\ & & \tfrac{1}{6}x_3 & + & x_4 & - & 2x_5 & = & 3 \\ & & \tfrac{1}{3}x_3 & & & - & x_5 & = & 66+z \end{array}$$

$$x_1,x_2,x_3,x_4,x_5 \geq 0$$

The problem remains in canonical form, but with basic variables x_1, x_2, and x_4. The associated basic solution $(4,23,0,3,0)$ is feasible, and the value of the objective function at this point is -66. Although the optimal value of z has not yet been attained, we have, as promised, moved to a basic feasible solution yielding a reduced value for z while maintaining the problem in canonical form.

Theorem 3.4.3. *In the problem of (3.4.1), assume that there is an index s such that $c_s < 0$ and that at least one $a_{is} > 0$, $i = 1,\ldots,m$. Suppose*

$$\frac{b_r}{a_{rs}} = \mathrm{Min}\left\{\frac{b_i}{a_{is}} : 1 \leq i \leq m \textit{ and } a_{is} > 0\right\}.$$

Then the problem can be put into canonical form with basic variables

$$x_1, x_2, \ldots, x_{r-1}, x_{r+1}, \ldots, x_m, x_s.$$

The value of the objective function at the associated basic feasible solution is

$$-z_0 + \frac{c_s b_r}{a_{rs}}$$

Proof. Consider the problem of (3.4.1) under the assumptions of the theorem. The coefficient $a_{rs} \neq 0$ (it is, in fact, positive), and so the term $a_{rs}x_s$ of the rth equation can be used as the pivot term in the pivot operation applied to the $m+1$ equations. By pivoting here, the system of constraints will be expressed in canonical form with basic variables $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_m, x_s$. The constant terms, b_i^* say, $i = 1,\ldots,m$, on the right side of the equations, become

$$b_i^* = b_i - \frac{a_{is}b_r}{a_{rs}}, \text{ for } i = 1,\ldots,m \text{ and } i \neq r \quad \text{and} \quad b_r^* = \frac{b_r}{a_{rs}} \tag{3.4.3}$$

Clearly $b_r^* \geq 0$. If $a_{is} \leq 0$ then, since $b_r \geq 0$ and $a_{rs} > 0$, $b_i^* \geq b_i \geq 0$. If $a_{is} > 0$ and $i \neq r$, by the choice of r, $b_i/a_{is} \geq b_r/a_{rs}$, and so $b_i \geq a_{is}b_r/a_{rs}$. Hence $b_i^* \geq 0$. Therefore the basic solution associated with these basic variables is feasible.

Now the objective function is given in (3.4.1) by the form $c_{m+1}x_{m+1} + \cdots + c_s x_s + \cdots + c_n x_n = z_0 + z$. The effect of the pivot operation on this equation will be to eliminate the x_s term from the equation, producing the equation

$$c_r^* x_r + c_{m+1}^* x_{m+1} + \cdots + c_{s-1}^* x_{s-1} + c_{s+1}^* x_{s+1} + \cdots + c_n^* x_n = z_0^* + z \tag{3.4.4}$$

with $z_0^* = z_0 - c_s b_r / a_{rs}$.

Thus the objective function is expressed in terms of only the new nonbasic variables and the value of this function at the new basic feasible solution is $-z_0 + c_s b_r / a_{rs}$. □

Notice the result of this pivot operation applied to the system of constraints and the objective function. The problem remains in canonical form with the original basic variable x_r replaced with the variable x_s. The value of the objective function at this new basic feasible solution is equal to the value $-z_0$ at the original basic feasible solution plus the quantity $c_s b_r / a_{rs}$. Since we have assumed that $c_s < 0$ and $a_{rs} > 0$, $c_s b_r / a_{rs}$ is less than or equal to zero, and is strictly less than zero if b_r is strictly positive. Thus, if $b_r > 0$, the pivot operation has left the system in canonical form at a basic feasible solution with a smaller value for the objective function. Let us assume for the time being that this is always the case, that any basic feasible solution to the system of constraints has no basic variable equal to zero. A basic solution with some basic variables equal to zero is called a *degenerate solution*, so we are assuming that all basic feasible solutions are nondegenerate.

Under this nondegeneracy hypothesis, Theorem 3.4.3 states that if at least one of the coefficients c_j, $m+1 \leq j \leq n$, is negative, say c_s, and if at least one of the coefficients a_{is}, $1 \leq i \leq m$, is positive, then a specific pivot operation leaves the problem in canonical form at a basic feasible solution that gives a reduced value for the objective function. Now we can continue. If the new coefficients of the objective function are all nonnegative, we are at the minimal value for the objective function, as Theorem 3.4.1 applies. If one of these coefficients is negative and if all of the coefficients of the associated variable are nonpositive in the constraint set, the objective function is unbounded below, as Theorem 3.4.2 applies. Otherwise, we can apply Theorem 3.4.3 again, driving to another basic feasible solution with an even smaller value for the objective function. Since at each step the value of the objective function is reduced (due to the nondegeneracy assumption), there can be no repetition of basic feasible solutions. The different values for the objective function guarantee that a particular basic feasible solution can appear at most once in the process (see Problem 10 of Section 3.2). Now there are at most a finite number of basic solutions, as there are only $\binom{n}{m} = n!/[m!(n-m)!]$ ways of selecting m basic variables from a set of n variables. Thus this process must eventually terminate. Either the minimum value of the objective function will be reached or the function will be proven to be unbounded.

This is the simplex method, with a proof, using the nondegeneracy hypothesis, that the process must terminate after a finite number of steps with either Theorems 3.4.1 or 3.4.2 applying. The nondegeneracy assumption is quite critical. If some basic feasible solutions were degenerate, the pivot operation of Theorem 3.4.3 applied in a row with $b_i = 0$ would leave the value of the objective function unchanged. After several steps of this, we would have no assurance that basic feasible solutions would not reappear, possibly causing the process to cycle indefinitely. In fact, examples of cycling have been constructed (see Appendix B). Thus, from a mathematical point of view, our proof of convergence of the process is inadequate. In Section 3.8 we will provide a complete proof that, for any linear programming problem, there exists a sequence of pivot operations that will drive the problem to completion.

From a practical point of view, however, a pleasant phenomenon occurs. The cliché "whatever can go wrong will go wrong" does not seem to apply. Although degeneracy occurs quite frequently in linear programming applications, very rarely will cycling occur. Simple rules such as those described below usually are sufficient to prevent cycling. The rules are certainly adequate to prevent cycling in the examples of this text (except, of course, for the example of Appendix B). Moreover, more precise rules for the selection of the pivoting term can be given that will guarantee that cycling does not occur (see Section 3.8).

We now summarize the steps of the simplex method, starting with the problem in canonical form.

1. If all $c_j \geq 0$, the minimum value of the objective function has been achieved (Theorem 3.4.1).
2. If there exists an s such that $c_s < 0$ and $a_{is} \leq 0$ for all i, the objective function is not bounded below (Theorem 3.4.2).
3. Otherwise, pivot (Theorem 3.4.3). To determine the pivot term:
   (a) Pivot in any column with a negative c_j term. If there are several negative c_j's, pivoting in the column with the smallest c_j may reduce the total number of steps necessary to complete the problem. Assume that we pivot in column s.
   (b) To determine the row of the pivot term, find that row, say row r, such that

   $$\frac{b_r}{a_{rs}} = \text{Min}\left\{\frac{b_i}{a_{is}} : a_{is} > 0\right\}$$

   Notice that here only those ratios b_i/a_{is} with $a_{is} > 0$ are considered. If the minimum of these ratios is attained in several rows, a simple rule such as choosing the row with the smallest index can be used to determine the pivoting row.
4. After pivoting, the problem remains in canonical form at a different basic feasible solution. Now return to step 1.

If the problem contains degenerate basic feasible solutions, proceed as above. These steps should still be adequate to drive the problem to completion.

Problem Set 3.4

1. Complete the problem of Example 3.4.3.
2. Solve the following using the ideas developed in this section.
   (a) Minimize $x_3 + x_4$ subject to

$$\begin{aligned} x_1 \qquad\qquad - x_4 &= 5 \\ x_2 + 2x_3 - 3x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(b) Minimize x_3 subject to the constraints of part (a).
(c) Minimize $x_3 - x_4$ subject to the constraints of part (a).
(d) Minimize $x_3 - x_4$ subject to

$$\begin{aligned} x_1 \qquad\qquad - x_4 &= 5 \\ x_2 + 2x_3 \qquad &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(e) Minimize $-x_3 + x_4$ subject to the constraints of part (d).
(f) Minimize $-x_3 + x_4$ subject to

$$\begin{aligned} x_1 \quad + \; x_3 - x_4 &= 0 \\ x_2 + 2x_3 \qquad &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(g) Minimize $-x_3 - x_4$ subject to the constraints of part (f).

3. Calculate the coefficient c_r^* in (3.4.4) on page 81. Can the variable removed from the basis at one step of the pivot operation return to the basis on the next step?

4. Using the form for the objective function given in (3.4.1) on page 78 and the coordinates of the new basic feasible solution given in (3.4.3) on page 81, by direct calculation show that the value of the objective function at the new basic feasible solution is as stated in Theorem 3.4.3.

5. Using (3.4.3) on page 81, determine when the pivot operation will go from a nondegenerate basic feasible solution to a degenerate basic feasible solution.

6. Suppose a problem is in canonical form and the associated basic feasible solution is degenerate, and x_1 is a basic variable with the value zero. The pivot operation is performed with the x_1 variable extracted from the basis. Describe the new basic feasible solution.

7. In Chapter 2 we saw linear programming problems with multiple optimal solution points. We do, however, have a uniqueness condition for problems in canonical form. Show that if a problem is driven to the canonical form in (3.4.1) and $c_j > 0$ for $m+1 \leq j \leq n$, then the minimal value $-z_0$ of the objective function is attained only at the point $(b_1, \ldots, b_m, 0, \ldots, 0)$.

8. Extend the formulas in the proof of Theorem 3.4.3 expressing the results of the pivot operation at the a_{rs} term. Show that for any $j \neq s$,

$$a_{ij}^* = a_{ij} - \frac{a_{is}a_{rj}}{a_{rs}}, \quad i \neq r$$

$$a_{rj}^* = \frac{a_{rj}}{a_{rs}}$$

$$c_j^* = c_j - \frac{c_s a_{rj}}{a_{rs}}$$

9. Consider the linear programming problem of (3.4.1). Suppose that the value of the function

$$z' = c'_{m+1}x_{m+1} + \cdots + c'_n x_n - z'_0$$

equals the value of the objective function

$$z = c_{m+1}x_{m+1} + \cdots + c_n x_n - z_0$$

in all solutions to the system of constraints of (3.4.1). Prove that

$$z_0' = z_0 \text{ and } c_j' = c_j \text{ for all } j, m+1 \leq j \leq n$$

Conclusion. Given a linear programming problem in canonical form with a specified set of basic variables, the coefficients in the expression for the objective function are unique.

3.5 THE SIMPLEX TABLEAU AND EXAMPLES

At each step of the simplex method, it is crucial to know only the basic variables and the values of the coefficients in the system of equations. To facilitate computation of a solution, at each step all we need do is record this information. This suggests a notation similar to the *detached coefficient* notation used for solving linear equations. We illustrate with the example of Section 3.3 [see equation (3.3.1)]. The problem, expressed in canonical form with basic variables x_2 and x_3, was, as in (3.3.4), to minimize z with

$$\begin{aligned}
-6x_1 \quad\quad + x_3 - 2x_4 + 2x_5 &= 6 \\
-3x_1 + x_2 \quad\quad + 6x_4 + 3x_5 &= 15 \\
5x_1 \quad\quad\quad\quad + 3x_4 - 2x_5 &= -21 + z \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}$$

This information is recorded in tableau form in Table 3.1.

The initial line of x's in the array simply labels the columns of the tableau with the variables of the problem. The first column identifies the basic variables. The first two rows correspond to the system of constraints, with the constant terms given in the last column. The last row corresponds to the equation defining the objective function, with the constant term on the right side of that equation in the last column and the z term suppressed from the tableau because it remains fixed throughout the simplex method.

We now apply the simplex method. As noted in Section 3.3, the -2 in the x_5 column of the last row indicates that we should pivot in that column. To determine the pivoting row, we compare the ratios b_i/a_{is} for $a_{is} > 0$, as in Theorem 3.4.3, and

Table 3.1

	x_1	x_2	x_3	x_4	x_5	
x_3	−6	0	1	−2	2	6
x_2	−3	1	0	6	3	15
	5	0	0	3	−2	−21

Table 3.2

	x_1	x_2	x_3	x_4	x_5	
x_3	-6	0	1	-2	②	6
x_2	-3	1	0	6	3	15
	5	0	0	3	-2	-21
x_5	-3	0	$\frac{1}{2}$	-1	1	3
x_2	⑥	1	$-\frac{3}{2}$	9	0	6
	-1	0	1	1	0	-15

Table 3.3

x_5	0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{7}{2}$	1	6
x_1	1	$\frac{1}{6}$	$-\frac{1}{4}$	$\frac{3}{2}$	0	1
	0	$\frac{1}{6}$	$\frac{3}{4}$	$\frac{5}{2}$	0	-14

find the row in which the minimum is attained. In this case $\frac{6}{2}$ is less than $\frac{15}{3}$ and, therefore, we should pivot at the 2 in the first row, replacing the basic variable x_3 with the variable x_5. The tableau representing the result of this pivot operation can be constructed from the present tableau by dividing the first row by 2 and then adding multiples of this row to the remaining rows in such a way as to generate zeros in the x_5 column. We illustrate in Table 3.2, placing this new tableau directly below the original tableau.

The second tableau represents the problem as stated in (3.3.6) on page 75. The associated basic feasible solution is $(0,6,0,0,3)$, and the value of the objective function at this point is the negative of the constant in the lower right-hand corner of the tableau, $-(-15) = 15$.

Pivoting now at the 6 in the x_1 column of the second row gives the tableau of Table 3.3. Since all the constants in the last row, excluding the -14, are nonnegative, the minimum value of the objective function has been attained. This value, $-(-14) = 14$, is attained at the basic feasible solution $(1,0,0,0,6)$, as can be read from the final tableau.

Hereafter the steps of the simplex method for any example will be recorded using this tableau notation. We emphasize that if at any time you are confused or bewildered by a statement based on the tableau presentation of a problem, simply translate the information in the tableau back into a clearly stated problem with the system of constraints and the objective function defined as usual, that is, "attach back" the variables. The tableau remains just a notation for a linear programming problem and the associated equations.

Table 3.4

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	3	2	0	1	0	0	60
x_5	-1	①	4	0	1	0	10
x_6	2	-2	5	0	0	1	50
	-2	-3	-3	0	0	0	0
x_4	⑤	0	-8	1	-2	0	40
x_2	-1	1	4	0	1	0	10
x_6	0	0	13	0	2	1	70
	-5	0	9	0	3	0	30
x_1	1	0	$-\frac{8}{5}$	$\frac{1}{5}$	$-\frac{2}{5}$	0	8
x_2	0	1	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	0	18
x_6	0	0	13	0	2	1	70
	0	0	1	1	1	0	70

Example 3.5.1. Maximize $2x_1 + 3x_2 + 3x_3$ subject to

$$\begin{aligned} 3x_1 + 2x_2 \quad\quad\; &\le 60 \\ -x_1 + x_2 + 4x_3 &\le 10 \\ 2x_1 - 2x_2 + 5x_3 &\le 50 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Introducing three slack variables and putting the problem into standard form gives the following:

$$\begin{array}{l} \text{Minimize } -2x_1 - 3x_2 - 3x_3 \\ \text{subject to} \\ 3x_1 + 2x_2 \qquad\qquad + x_4 \qquad\qquad\quad = 60 \\ -x_1 + x_2 + 4x_3 \qquad\quad + x_5 \qquad = 10 \\ 2x_1 - 2x_2 + 5x_3 \qquad\qquad\quad + x_6 = 50 \\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{array}$$

The system of constraints for this problem is in canonical form with basic variables x_4, x_5, and x_6, the associated basic solution, $(0,0,0,60,10,50)$, is feasible, and the objective function is written in terms of the nonbasic variables. Thus the simplex method can be initiated. Table 3.4 gives the resulting tableaux.

Note that the first pivot could have been made in either the first, second, or third column. From the last tableau we see that, for the problem as stated in standard form, the minimal value of the objective function is -70, and this value is attained at the point $(8,18,0,0,0,70)$. Since the original problem was a maximization problem with no slack variables, the optimal value for the original objective function is 70 and is attained at the point $(8,18,0)$.

Table 3.5

	x_1	x_2	x_3	x_4	x_5	
x_4	1	1	-2	1	0	7
x_5	-3	①	2	0	1	3
	0	-2	-1	0	0	0
x_4	④	0	-4	1	-1	4
x_2	-3	1	2	0	1	3
	-6	0	3	0	2	6
x_1	1	0	-1	$\frac{1}{4}$	$-\frac{1}{4}$	1
x_2	0	1	-1	$\frac{3}{4}$	$\frac{1}{4}$	6
	0	0	-3	$\frac{3}{2}$	$\frac{1}{2}$	12

Example 3.5.2. Maximize $2x_2 + x_3$ subject to

$$\begin{aligned} x_1 + x_2 - 2x_3 &\leq 7 \\ -3x_1 + x_2 + 2x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The standard form of the problem is

$$\begin{aligned} &\text{Minimize } -2x_2 - x_3 \\ &\text{subject to} \\ &\quad x_1 + x_2 - 2x_3 + x_4 \qquad\quad = 7 \\ &-3x_1 + x_2 + 2x_3 \qquad + x_5 = 3 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

This problem is in canonical form with basic variables x_4 and x_5, and the steps of the simplex algorithm are displayed in Table 3.5. The three negative entries in the third column of the previous tableau indicate that the objective function is unbounded below.

Example 3.5.3. Finally, we consider the problem of

$$\begin{aligned} &\text{Minimizing } -4x_1 + x_2 + 30x_3 - 11x_4 - 2x_5 + 3x_6 \\ &\text{subject to} \\ &-2x_1 \qquad + 6x_3 + 2x_4 \qquad - 3x_6 + x_7 = 20 \\ &-4x_1 + x_2 + 7x_3 + \ x_4 \qquad - \ x_6 \qquad = 10 \\ &\qquad\qquad - 5x_3 + 3x_4 + x_5 - \ x_6 \qquad = 60 \\ &x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

The system of constraints, as given, is in canonical form with basic variables x_7, x_2, and x_5, and the associated basic solution, $(0, 10, 0, 0, 60, 0, 20)$, is feasible. However,

Table 3.6

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_7	-2	0	6	2	0	-3	1	20
x_2	-4	1	7	①	0	-1	0	10
x_5	0	0	-5	3	1	-1	0	60
	0	0	13	-6	0	2	0	110
x_7	⑥	-2	-8	0	0	-1	1	0
x_4	-4	1	7	1	0	-1	0	10
x_5	12	-3	-26	0	1	2	0	30
	-24	6	55	0	0	-4	0	170
x_1	1	$-\frac{1}{3}$	$-\frac{4}{3}$	0	0	$-\frac{1}{6}$	$\frac{1}{6}$	0
x_4	0	$-\frac{1}{3}$	$\frac{5}{3}$	1	0	$-\frac{5}{3}$	$\frac{2}{3}$	10
x_5	0	1	-10	0	1	④	-2	30
	0	-2	23	0	0	-8	4	170
x_1	1	$-\frac{7}{24}$	$-\frac{7}{4}$	0	$\frac{1}{24}$	0	$\frac{1}{12}$	$\frac{5}{4}$
x_4	0	$\frac{1}{12}$	$-\frac{5}{2}$	1	$\frac{5}{12}$	0	$-\frac{1}{6}$	$\frac{45}{2}$
x_6	0	$\frac{1}{4}$	$-\frac{5}{2}$	0	$\frac{1}{4}$	1	$-\frac{1}{2}$	$\frac{15}{2}$
	0	0	3	0	2	0	0	230

the expression for the objective function contains the basic variables x_2 and x_5. By subtracting the second equation and adding twice the third equation to the equation

$$-4x_1 + x_2 + 30x_3 - 11x_4 - 2x_5 + 3x_6 = z$$

we have

$$13x_3 - 6x_4 + 2x_6 = 110 + z$$

Using this expression to define the objective function, the problem is in canonical form with basic variables x_7, x_2, and x_5, and the simplex method can be initiated. The corresponding tableaux are given in Table 3.6. As can be seen, the minimal value of the objective function is -230 and is attained at the point $(\frac{5}{4}, 0, 0, \frac{45}{2}, 0, \frac{15}{2}, 0)$. Note the presence of degeneracy in the second and third steps.

Problem Set 3.5

1. Each of the following tableaux corresponds to a linear programming problem in canonical form with three equality constraints, an objective function to be minimized, seven nonnegative variables $x_1, \ldots, x_7$, and with variables x_5, x_3, x_1 serving as basic variables. For each, either (i) the solution of the problem can be determined from the given tableau or (ii) one or more iterations of the simplex algorithm are necessary to complete the problem. If (i), state the complete resolution of the problem; if (ii), determine all valid pivot points for the tableau.

(a)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	5	0	3	1	−1	8	39
x_3	0	6	1	−1	0	0	−6	10
x_1	1	9	0	8	0	−3	4	88
	0	6	0	−4	0	2	3	$-75+z$

(b)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	5	0	−3	1	−1	8	39
x_3	0	6	1	1	0	−1	−6	10
x_1	1	9	0	−8	0	−3	4	88
	0	6	0	4	0	2	0	$-75+z$

(c)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	5	0	−3	1	−1	8	3
x_3	0	6	1	1	0	0	−6	2
x_1	1	9	0	−8	0	−3	4	0
	0	−6	0	0	0	0	3	$-75+z$

(d)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	5	0	−3	1	−1	8	3
x_3	0	6	1	1	0	0	−6	2
x_1	1	9	0	−8	0	−3	4	1
	0	−6	0	0	0	−2	3	$-75+z$

(e)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	5	0	−3	1	1	8	60
x_3	0	6	1	−1	0	0	−6	30
x_1	1	9	0	−8	0	−3	7	50
	0	−6	0	0	0	−2	−3	$-75+z$

(f)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	−5	0	−3	1	−1	8	39
x_3	0	−6	1	−1	0	0	−6	0
x_1	1	9	0	−8	0	−3	4	88
	0	6	0	0	0	2	3	$-75+z$

2. For each of the following, put the problem into canonical form, set up the initial tableau, and solve by hand using the simplex method. At most, two pivots should be required for each. Along the way, objective functions requiring some initial adjustments and unbounded objective functions should be encountered.

(a) Minimize $2x_1 + 4x_2 - 4x_3 + 7x_4$
subject to

$$\begin{array}{rcrcrcrcr} 8x_1 & - & 2x_2 & + & x_3 & - & x_4 & \leq & 50 \\ 3x_1 & + & 5x_2 & & & + & 2x_4 & \leq & 150 \\ x_1 & - & x_2 & + & 2x_3 & - & 4x_4 & \leq & 100 \end{array}$$

$x_1, x_2, x_3, x_4 \geq 0$

(b) Maximize $x_1 + 2x_2 - x_3$
subject to

$$\begin{array}{rcrcrcr} & & x_2 & + & 4x_3 & \leq & 36 \\ 5x_1 & - & 4x_2 & + & 2x_3 & \leq & 60 \\ 3x_1 & - & 2x_2 & + & x_3 & \leq & 24 \end{array}$$

$x_1, x_2, x_3 \geq 0$

(c) Minimize $-5x_1 + 4x_2 + x_3$
subject to

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & 3x_3 & \leq & 8 \\ & & 2x_2 & - & 2x_3 & \leq & 7 \\ -x_1 & - & 2x_2 & + & 4x_3 & \leq & 6 \end{array}$$

$x_1, x_2, x_3 \geq 0$

(d) Maximize $9x_2 + 2x_3 - x_5$
subject to

$$\begin{array}{rcrcrcrcrcrcr} x_1 & - & 3x_2 & & & - & 4x_4 & & & + & 2x_6 & = & 60 \\ & & 2x_2 & & & - & x_4 & - & x_5 & + & 4x_6 & = & -20 \\ & & x_2 & + & x_3 & & & & & + & 3x_6 & = & 10 \end{array}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

(e) Maximize $x_1 + 12x_2 + 9x_3$
subject to

$$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & 6x_3 & \leq & 20 \\ 2x_1 & + & 6x_2 & + & 3x_3 & \leq & 30 \\ 6x_1 & & & + & 2x_3 & \leq & 16 \end{array}$$

$x_1, x_2, x_3 \geq 0$

(f) Minimize $x_3 - x_4$
subject to

$$\begin{array}{rrrcrcrcr} x_1 & & & - & 3x_4 & + & x_5 & = & 1 \\ & x_2 & & + & 6x_4 & - & 5x_5 & = & 6 \\ & & x_3 & - & 3x_4 & + & 2x_5 & = & 5 \end{array}$$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

For the remaining problems, the use of the software LP Assistant, as described in Appendix D, is strongly encouraged. The program facilitates considerably the implementation of the simplex method. The user needs to enter a valid initial tableau and appropriate pivots points, and needs to recognize a final tableau and interpret the results, but the machine completes the arithmetic of each pivot step.

3. Solve. Maximize $x_4 - x_5$
subject to

$$\begin{aligned} x_1 \quad\quad &+ x_4 - 2x_5 = 1 \\ x_2 \quad &+ x_4 \quad\quad = 6 \\ x_3 &+ 2x_4 - 3x_5 = 4 \\ x_1, x_2, x_3&, x_4, x_5 \geq 0 \end{aligned}$$

Note that in this example a variable removed from the basis in one step of the pivot operation eventually returns to the basis. Compare with Problem 3 of Section 3.4.

4. Solve. Maximize $10x_3 + 3x_4$
subject to

$$\begin{aligned} x_1 \quad &+ 10x_3 + 2x_4 = 20 \\ x_2 &- \quad x_3 + \quad x_4 = 12 \\ x_1, x_2&, x_3, x_4 \geq 0 \end{aligned}$$

(If in your first iteration you put x_3 into the basis, you will have an example of a variable inserted into the basis in one step of the simplex algorithm being removed from the basis in the very next step.)

5. Consider the problem of Example 3.5.3. The minimum value of the objective function is -230 and is attained at $(\frac{5}{4}, 0, 0, \frac{45}{2}, 0, \frac{15}{2}, 0)$. However, this optimal value is attained at other solution points to the system of constraints.

(a) The previous tableau for the solution to this problem suggests that optimal basic feasible solutions exist with either x_2 or x_7 in the basis. Why?

(b) Use the previous tableau to determine an optimal basic feasible solution with x_7 in the basis.

(c) Find an optimal solution with x_2 in the basis.

6. For each of the following, determine two distinct basic feasible solutions at which the optimal value of the objective function is attained.

(a) Maximize $4x_1 + 12x_2 + 8x_3$
subject to

$$\begin{aligned} 3x_1 + 2x_2 - 6x_3 &\leq 20 \\ 3x_1 + 6x_2 + 4x_3 &\leq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(b) Minimize $x_1 - 3x_2 - 6x_3$
subject to
$$\begin{aligned} 2x_1 - x_2 + x_3 + x_4 &\leq 60 \\ 3x_1 + 4x_2 + 2x_3 - 2x_4 &\leq 150 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

7. Consider the problem of Example 3.5.2.

   (a) Show that any point of the form $(t,0,t)$, for $t \geq 0$, is a feasible solution.
   (b) Using this, show that the objective function is unbounded.

8. Compute the solution to Problem 11 of Section 2.3.

9. Compute the solution to Problem 7 of Section 2.6.

10. Compute the solution to Problem 5 of Section 2.6

3.6 ARTIFICIAL VARIABLES

As we have seen, many linear programming problems can be put into canonical form with little or no effort. For example, the addition of slack variables with positive coefficients can provide the basic variables necessary for the initial basic feasible solution. On the other hand, the system of constraints for many other problems contains no obvious basic feasible solutions. Problems of this type occur, for example, in production models involving output requirements and therefore ($\geq$) inequalities in the constraint set, such as we saw in Example 2.3.4 on page 24, or in transportation problems involving fixed demands and therefore equalities in the constraint set, such as in Example 2.4.1 on page 34. In fact, in any application of linear programming to a real-world problem, it would be rare to find the original formulation of the problem in canonical form.

What must be developed is a technique for determining an initial basic feasible solution for an arbitrary system of equations. This technique must also be capable of handling problems having no feasible solution. Such a problem could arise, for example, in a model containing an error in formulation or in a complicated production model where it is not obvious that the various output requirements can be met with the limited resources available. In this section we will introduce such a technique; in the next section we will discuss some of the complications that can occur.

The basic idea behind the method used to find an initial basic feasible solution is simple. We introduce into the problem a sufficient number of variables, called *artificial variables*, to put the system of constraints into canonical form with these variables as the basic variables. Then we apply the simplex method, not to the objective function of the original problem, but to a new function defined in such a way that its minimal value is attained at a feasible solution to the original problem. Thus the method of the previous three sections applied to this new function drives the original problem to a basic feasible solution.

Consider the standard linear programming problem of (3.1.1) of finding a nonnegative solution to the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{3.6.1}$$

that minimizes the function $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$. By multiplication of an equation by (-1) if necessary, we may assume that all the constant terms b_i, $i = 1,\ldots,m$, are nonnegative. Now introduce into the system of constraints m new variables, x_{n+1}, $\ldots$, x_{n+m}, called *artificial variables*, one to each equation. The resulting system is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} \qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad + x_{n+2} \qquad &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \qquad\qquad + x_{m+n} &= b_m \end{aligned} \tag{3.6.2}$$

Note that this system is in canonical form with basic variables x_{n+1}, $\ldots$, x_{m+n}, and that the associated basic solution is feasible, since we have assumed that the b_i's are nonnegative.

Now consider the problem of determining the minimal value of the function $w = x_{n+1} + x_{n+2} + \cdots + x_{n+m}$ on the set of all nonnegative solutions to the system of equations in (3.6.2). Since all variables are nonnegative, w can never be negative. The function w would assume the value zero at any feasible solution to (3.6.2) in which all the artificial variables are at zero level. Thus the simplex method applied to this function should replace the artificial variables as basic variables with the variables from the original problem and will hopefully drive the system in (3.6.2) into canonical form with basic variables from the original set x_j, $j = 1,\ldots,n$. The value of w at the associated basic feasible solution would be zero, its minimal value, and the simplex method could then be initiated on the original problem as stated in (3.6.1). Furthermore, if the system of constraints in (3.6.1) does have at least one feasible solution, the system in (3.6.2) must have feasible solutions in which all the artificial variables equal zero. In this case the minimal value of w would be, in fact, zero. Thus, when applying the simplex method to the function w, if we reach a step at which we can pivot no more but the associated value of w is greater than zero, we can conclude that the original problem has no feasible solutions.

Before we present examples, some remarks of a technical nature are in order. First, before the simplex method can be applied to the function $w = x_{n+1} + x_{n+2} + \cdots + x_{n+m}$, the problem must be in canonical form. The system of constraints in (3.6.2) is in canonical form with the artificial variables as basic variables and the associated basic solution is feasible, but the function w is not expressed in terms of only the nonbasic variables. To rectify this, we subtract from the equation defining w each constraining equation containing an artificial variable. (In the general problem above, artificial variables have been introduced into every constraint. However, this need not always be the case. In some instances, some of the original problem variables may be used in the initial basic variable set. An example will be seen in Example 3.6.2 shortly.)

Second, if the pivot operations dictated by the problem of minimizing w are also simultaneously performed on the equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = z$ which defines the original objective function, this function will be expressed in terms of nonbasic variables at each step. Thus, if an initial basic feasible solution is found for the original problem, the simplex method can be initiated immediately on z. Therefore we incorporate this z equation into the notation and operations of the problem of minimizing w.

In the sum, the first step in solving the general problem of (3.6.1) is to consider the problem of minimizing w with

$$\begin{array}{lllllllll} a_{11}x_1 & + & a_{12}x_2 & + \ldots + & a_{1n}x_n & + x_{n+1} & & & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \ldots + & a_{2n}x_n & & + x_{n+2} & & = b_2 \\ \vdots & & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + \ldots + & a_{mn}x_n & & & + x_{m+n} & = b_m \\ c_1x_1 & + & c_2x_2 & + \ldots + & c_nx_n & & & & = z \\ d_1x_1 & + & d_2x_2 & + \ldots + & d_nx_n & & & & = w_0 + w \end{array} \tag{3.6.3}$$

where $d_j = -(a_{1j} + a_{2j} + \cdots + a_{mj})$ and $w_0 = -(b_1 + b_2 + \cdots + b_m)$.

Example 3.6.1. Consider the problem to

$$\begin{array}{l} \text{Minimize } 2x_1 - 3x_2 + x_3 + x_4 \\ \text{subject to} \\ x_1 - 2x_2 - 3x_3 - 2x_4 = 3 \\ x_1 - x_2 + 2x_3 + x_4 = 11 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array} \tag{3.6.4}$$

Introducing artificial variables x_5 and x_6, we now instead consider the problem of minimizing w where

$$\begin{array}{l} x_1 - 2x_2 - 3x_3 - 2x_4 + x_5 = 3 \\ x_1 - x_2 + 2x_3 + x_4 + x_6 = 11 \\ 2x_1 - 3x_2 + x_3 + x_4 = z \\ x_5 + x_6 = w \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array} \tag{3.6.5}$$

Subtracting the first two equations from the w equation gives the system

$$\begin{array}{l} x_1 - 2x_2 - 3x_3 - 2x_4 + x_5 = 3 \\ x_1 - x_2 + 2x_3 + x_4 + x_6 = 11 \\ 2x_1 - 3x_2 + x_3 + x_4 = z \\ -2x_1 + 3x_2 + x_3 + x_4 = -14 + w \end{array}$$

Table 3.7

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	1	−2	−3	−2	1	0	3
x_6	1	−1	2	1	0	1	11
	2	−3	1	1	0	0	0
	−2	3	1	1	0	0	−14

Table 3.8

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	①	−2	−3	−2	1	0	3
x_6	1	−1	2	1	0	1	11
	2	−3	1	1	0	0	0
	−2	3	1	1	0	0	−14
x_1	1	−2	−3	−2	1	0	3
x_6	0	1	⑤	3	−1	1	8
	0	1	7	5	−2	0	−6
	0	−1	−5	−3	2	0	−8
x_1	1	$-\frac{7}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{39}{5}$
x_3	0	$\frac{1}{5}$	1	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{8}{5}$
	0	$-\frac{2}{5}$	0	$\frac{4}{5}$	$-\frac{3}{5}$	$-\frac{7}{5}$	$-\frac{86}{5}$
	0	0	0	0	1	1	0

This information can be recorded in tableau form by simply augmenting the notation of the previous section (see Table 3.7). The last row corresponds to the w equation, with the w suppressed from the notation. Now the simplex method is initiated, with the entries in the last row determining the pivoting column at each step. The second to last row, the z row, is operated on at each pivot operation but is otherwise ignored for the time being. Table 3.8 gives the resulting tableaux.

Thus the minimal value of w is 0, and one point at which this value is attained is $(\frac{39}{5},0,\frac{8}{5},0,0,0)$. Since this point is a solution to the system of constraints in (3.6.5) and has as its last two coordinates zero, $(\frac{39}{5},0,\frac{8}{5},0)$ is a basic feasible solution to the system in (3.6.4), and the data for the tableau corresponding to the original problem expressed in canonical form with basic variables x_1 and x_3 are contained in the last tableau. In fact, translating these data back into equation form gives the following system, equivalent to (3.6.4).

$$\begin{aligned} x_1 - \tfrac{7}{5}x_2 \qquad\quad - \tfrac{1}{5}x_4 &= \tfrac{39}{5} \\ \tfrac{1}{5}x_2 + x_3 + \tfrac{3}{5}x_4 &= \tfrac{8}{5} \\ -\tfrac{2}{5}x_2 \qquad\quad + \tfrac{4}{5}x_4 &= -\tfrac{86}{5} + z \end{aligned}$$

Table 3.9

	x_1	x_2	x_3	x_4	
x_1	1	$-\frac{7}{5}$	0	$-\frac{1}{5}$	$\frac{39}{5}$
x_3	0	$\enclose{circle}{\frac{1}{5}}$	1	$\frac{3}{5}$	$\frac{8}{5}$
	0	$-\frac{2}{5}$	0	$\frac{4}{5}$	$-\frac{86}{5}$
x_1	1	0	7	4	19
x_2	0	1	5	3	8
	0	0	2	2	-14

The second stage of the problem, the application of the simplex process to the problem of minimizing z, can be initiated immediately (Table 3.9). The minimal value of z is 14 and is attained at the point $(19,8,0,0)$.

The above computational procedure can be streamlined somewhat. First, there is no need to make a formal break in the tableau notation when passing from the first stage of a linear programming problem, the minimization of the w function, to the second stage, the minimization of the z function. Once a basic feasible solution to the original problem has been found, the w row of the augmented tableau notation can be dropped and the problem continued directly using the z row.

Second, once an artificial variable is extracted from the basis, there is no need to reenter it in any future step. To see this, consider the above example after the first pivot operation. The data of the first two rows of the second tableau of Table 3.8 correspond to the following two equations:

$$\begin{aligned} x_1 - 2x_2 - 3x_3 - 2x_4 + x_5 \quad &= 3 \\ x_2 + 5x_3 + 3x_4 - x_5 + x_6 &= 8 \end{aligned} \tag{3.6.6}$$

Setting x_5, the artificial variable removed from the basis in the first iteration, equal to zero yields the system of equations

$$\begin{aligned} x_1 - 2x_2 - 3x_3 - 2x_4 \quad &= 3 \\ x_2 + 5x_3 + 3x_4 + x_6 &= 8 \end{aligned} \tag{3.6.7}$$

a system equivalent to the constraints of (3.6.5) with $x_5 = 0$, that is, the system of equations

$$\begin{aligned} x_1 - 2x_2 - 3x_3 - 2x_4 \quad &= 3 \\ x_1 - x_2 + 2x_3 + x_4 + x_6 &= 11 \end{aligned} \tag{3.6.8}$$

Now the constraints of the original problem (3.6.4) have feasible solutions if and only if (3.6.8) has feasible solutions with $x_6 = 0$ if and only if (3.6.7) has feasible solutions with $x_6 = 0$. Thus, if (3.6.4) has feasible solutions, the simplex algorithm applied the problem of minimizing the function "w" $= x_6$ subject to the constraints of (3.6.7) would drive this modified w function to zero using only the variables of (3.6.7). (Notice that to apply the algorithm to the function "w" $= x_6$ subject to the constraints of (3.6.7), the basic variables of (3.6.7), x_1 and x_6, would first need to be extracted from the expression for the objective function. Thus the second equation of (3.6.7) would be subtracted from this expression; the resulting form is exactly that of the bottom row of the second tableau of Table 3.8, with x_5 set equal to zero.) Hence the artificial variable x_5 need never return to the basis after the first iteration. As a result, in applying the simplex algorithm, it is never necessary to use the information in the artificial variable columns of the tableau, and so these data need not be calculated at each pivot step.

Example 3.6.2. Minimize $x_1+x_2+x_3=z$ subject to

$$\begin{aligned}
-x_1 + 2x_2 + x_3 &\le 1\\
-x_1 \qquad + 2x_3 &\ge 4\\
x_1 - x_2 + 2x_3 &= 4\\
x_1,x_2,x_3 &\ge 0
\end{aligned}$$

Adding two slack variables, the problem in standard form becomes

$$\begin{aligned}
&\text{Minimize } x_1+x_2+x_3=z\\
&\text{subject to}\\
&-x_1 + 2x_2 + x_3 + x_4 \qquad = 1\\
&-x_1 \qquad + 2x_3 \qquad - x_5 = 4\\
&x_1 - x_2 + 2x_3 \qquad\qquad = 4\\
&x_1,x_2,x_3,x_4,x_5 \ge 0
\end{aligned}$$

Note that the x_4 variable can serve as a basic variable. Thus it is sufficient to add only two artificial variables, say x_6 and x_7, to the problem and at the first stage minimize the function $w=x_6+x_7$. The problem is then

$$\begin{aligned}
-x_1 + 2x_2 + x_3 + x_4 \qquad\qquad\qquad &= 1\\
-x_1 \qquad + 2x_3 \qquad - x_5 + x_6 \qquad &= 4\\
x_1 - x_2 + 2x_3 \qquad\qquad\qquad + x_7 &= 4\\
x_1 + x_2 + x_3 \qquad\qquad\qquad\qquad &= z\\
x_6 + x_7 &= w
\end{aligned}$$

Subtracting the second and third equations from the w equation gives the equation $x_2-4x_3+x_5=-8+w$. Now the expression for w does not contain the initial basic variables x_4, x_6, and x_7, and the simplex method can be initiated. The resulting tableaux are given in Table 3.10. The minimal value for the function $w=x_6+x_7$ is $\frac{4}{3}$, and this value is attained at the point $(\frac{2}{3},0,\frac{5}{3},0,0,\frac{4}{3},0)$. Therefore we can conclude that the original problem has no feasible solution.

Problem Set 3.6

Note: Again the use of the LP Assistant software is strongly recommended. The program provides easy designation of artificial variables and automatically computes the relevant w-function data into the working tableau.

Table 3.10

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	-1	2	①	1	0	0	0	1
x_6	-1	0	2	0	-1	1	0	4
x_7	1	-1	2	0	0	0	1	4
	1	1	1	0	0			0
	0	1	-4	0	1			-8
x_3	-1	2	1	1	0			1
x_6	1	-4	0	-2	-1			2
x_7	③	-5	0	-2	0			2
	2	-1	0	-1	0			-1
	-4	9	0	4	1			-4
x_3	0	$\frac{1}{3}$	1	$\frac{1}{3}$	0			$\frac{5}{3}$
x_6	0	$-\frac{7}{3}$	0	$-\frac{4}{3}$	-1			$\frac{4}{3}$
x_1	1	$-\frac{5}{3}$	0	$-\frac{2}{3}$	0			$\frac{2}{3}$
	0	$\frac{7}{3}$	0	$\frac{1}{3}$	0			$-\frac{7}{3}$
	0	$\frac{7}{3}$	0	$\frac{4}{3}$	1			$-\frac{4}{3}$

1. Using the technique described in this section, find solutions with nonnegative coordinates to the following systems of equations.

(a)
$$\begin{aligned} x_1 - x_2 \phantom{{}- x_3} &= 1 \\ 2x_1 + x_2 - x_3 &= 3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 \phantom{{}- x_3} &= 1 \\ 2x_1 + x_2 - x_3 &= 3 \end{aligned}$$

2. Solve the following.

(a) Minimize $2x_1 + 2x_2 - 5x_3$
subject to
$$\begin{aligned} 3x_1 + 2x_2 - 4x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(b) Minimize $x_1 - 3x_3$
subject to
$$\begin{aligned} x_1 + 2x_2 - x_3 &\leq 6 \\ x_1 - x_2 + 3x_3 &= 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(c) Minimize $x_1 + x_2 - x_4$
subject to
$$\begin{aligned} 4x_1 + x_2 + x_3 + 4x_4 &= 8 \\ x_1 - 3x_2 + x_3 + 2x_4 &= 16 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(d) Maximize $3x_1 - x_2$
subject to
$$\begin{aligned} x_1 - x_2 &\leq 3 \\ 2x_1 &\leq x_2 \\ x_1 + x_2 &\geq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(e) Maximize $x_1 + 2x_2 + 3x_3 + 4x_4$
subject to
$$\begin{aligned} x_1 \qquad + x_3 - 4x_4 &= 2 \\ x_2 - x_3 + 3x_4 &= 9 \\ x_1 + x_2 - 2x_3 - 3x_4 &= 21 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(f) Minimize $8x_1 - 2x_2 - x_3 - 6x_4$
subject to
$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 12 \\ -2x_1 + 3x_2 \qquad + 2x_4 &= 42 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(g) Minimize $3x_1 - x_2 + 2x_3 + 5x_4 + 6x_5$
subject to
$$\begin{aligned} 12x_1 - 3x_2 + 5x_3 - 2x_4 + 4x_5 &= 100 \\ 8x_1 - 2x_2 - 4x_3 \qquad + 5x_5 &= 150 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

3. Using a combination of birdseed mixtures A, B, and C, a blend of minimum cost which is at least 20% thistle and 30% corn is desired. Given the data which follow, determine the percentages of each of the mixtures in the final blend.

	% Thistle	*% Corn*	*Cost (cents/lb)*
A	25	40	57
B	0	30	13
C	10	15	20

4. Consider the tableaux for the first stage of the problem discussed in Example 3.6.1. The very last row, the *w* row after the second pivot step in Table 3.8, pivot step to determine both the exiting and entering variables, when there is more than one eligible variable, use the variable with the smallest index.

Although these procedures solve the cycling problem in theory, cycling in practice is another question. Various factors influencing cycling can be involved in a computer implementation of the simplex algorithm, such as roundoff errors, special pivoting rules, data scaling, and built-in perturbation techniques; and in fact, some linear programming problems have caused cycling in some programmed versions of the algorithm (see, e.g., [10]). However, the issue of cycling in practice is just part of the broader question of the efficiency of a given solution algorithm being implemented on a particular computer system to resolve the specific class of problems under consideration.

Problem Set 3.8

1. Prove Lemma 3.8.1.
2. Prove Lemma 3.8.2. *Hint.* Consider the effect or noneffect of these pivot operations on the b_i column and the c_j row.
3. Prove Corollary 3.8.1. Note that Theorem 3.8.1 applies only to a problem presented in canonical form.
4. True or false: Suppose the simplex method is applied to a linear programming problem presented in canonical form and that, at each step, there is at most one term that could serve as a pivot term. Then for this problem, cycling is impossible.
5. True or false: Given a linear programming problem with $n = m + 1$ and presented in canonical form, at most one step in the simplex method is necessary to drive the process to termination.
6. Using Lemma 3.8.2, solve the linear programming problem of:
   (a) Example 3.5.1, but with the constant terms 60, 10, and 50 replaced with zeros.
   (b) Example 3.5.2, but with the constant terms 7 and 3 replaced with zeros.
7. True or false: Given a linear programming problem with all the constant terms of the system of constraints equal to zero, either the objective function is unbounded or it attains its optimal value at the point zero.

3.9 LINEAR PROGRAMMING AND CONVEXITY

In Section 2.2 we considered a linear programming problem involving only two variables. We were able to graph the set of feasible solutions to the set of constraints (Figure 2.3) and, by a geometric argument, show that the optimal value of the linear objective function must be attained at a corner or vertex to this solution set. This result generalizes, as suggested at the end of Section 3.2. In this section we will first define the concept of convexity and show that the solution set to a system of

Figure 3.4

equations and inequalities is convex. Then we will define the concept of a vertex of a convex set and relate the basic feasible solutions of a system of constraints to the vertices of the solution set to this system. The corollary of the previous section will then give directly the generalization of the above result.

Only the concept of convexity will be used later in the book, and then not until Section 8.3 and Chapter 10. We present these ideas here primarily to initiate an appreciation of some of the geometry underlying the linear programming problem.

For two points P and Q in $\mathbb{R}^n$, the *line segment* between P and Q is that set of points in $\mathbb{R}^n$ of the form $tP+(1-t)Q$ for $0 \leq t \leq 1$ (see Problem 1). A subset S of $\mathbb{R}^n$ is said to be *convex* if, for any two points of S, the line segment between these two points is also in S.

Example 3.9.1. Of the six subsets of $\mathbb{R}^2$ shown in Figure 3.4, each of the three on the left is convex, while none of the three on the right is convex.

Example 3.9.2. Let $S = \{(x_1,x_2) \in \mathbb{R}^2 : x_1+x_2 \geq 2\}$. Then S is convex, a fact obvious from a graph of S. To prove this algebraically using only our definitions, take any two points $P=(p_1,p_2)$ and $Q=(q_1,q_2)$ in S. Then $p_1+p_2 \geq 2$ and $q_1+q_2 \geq 2$. Take any point

$$tP+(1-t)Q = (tp_1+(1-t)q_1, tp_2+(1-t)q_2), \text{ with } 0 \leq t \leq 1$$

on the line segment between P and Q. We have

$$\begin{aligned} tp_1+(1-t)q_1+tp_2+(1-t)q_2 &= t(p_1+p_2)+(1-t)(q_1+q_2) \\ &\geq 2t+2(1-t) \\ &= 2 \end{aligned}$$

using the fact that t and $1-t$ are nonnegative. Thus $tP+(1-t)Q$ is in S, and we have an algebraic proof that S is convex.

The set of feasible solutions to a linear programming problem is convex, since it is the intersection of a collection of hyperplanes and half-spaces. We state these results in the following, leaving the proofs of the theorems for the reader.

Definition 3.9.1. A subset of $\mathbb{R}^n$ of the form

$$X = \{(x_1,\dots,x_n) : a_1x_1+a_2x_2+\cdots+a_nx_n = b\}$$

for constants $a_1,a_2,\dots,a_n$ and b is called a *hyperplane*.

A subset of the form

$$X = \{(x_1,\ldots,x_n) : a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b\}$$

for constants $a_1, a_2, \ldots, a_n$ and b is called a *half-space.*

Theorem 3.9.1. *A half-space is convex.*

Theorem 3.9.2. *The intersection of two convex sets is convex.*

Corollary 3.9.1. *The set of feasible solutions to a linear programming problem is convex.*

Intuitively, the corners or vertices of a convex set are those points of the set that do not lie on the interior of a line segment contained in the set. This suggests the following.

Definition 3.9.2. A point P of a convex set S is a *vertex* of S if P is not the midpoint of a line segment connecting two other points of S.

Example 3.9.3. For the three convex figures of Example 3.9.1, the line segment has two vertices (the two end points), the triangle has three (the three corners), and the home plate has five.

Theorem 3.9.3. *Let S be the set of feasible solutions to the system of constraints of a linear programming problem in a standard form. Then any basic feasible solution to the problem is a vertex of S.*

Proof. Let X be a basic feasible solution, and suppose the first m variables are the basic variables, with n the total number of variables. Assume $X = (P+Q)/2$, where $P = (p_1,\ldots,p_n)$ and $Q = (q_1,\ldots,q_n)$ are in S. Then

$$\begin{aligned} X &= (x_1,\ldots,x_n,0,\ldots,0) \\ &= \frac{1}{2}(p_1+q_1,\ldots,p_m+q_m,p_{m+1}+q_{m+1},\ldots,p_n+q_n). \end{aligned}$$

Since all the coordinates of P and Q are nonnegative,

$$p_j = q_j = 0 \text{ for } j = m+1,\ldots,n$$

But there is only one basic feasible solution, X, with all these coordinates equal to zero (see Problem 10 of Section 3.2). Thus $P = Q = X$. Hence X is a vertex of S. □

Corollary 3.9.2. *If the objective function of a linear programming problem has a finite optimal value, this value is assumed by at least one vertex of the set of feasible solutions to the system of constraints.*

Proof. This follows directly from Theorem 3.9.3 and Corollary 3.8.1. □

In the simplex algorithm we move from basic feasible solution to basic feasible solution by replacing at each step one variable in the basis. From Theorem 3.9.3, we see that we are simply moving from vertex to vertex in the convex set of feasible solutions to the system of constraints. In fact, since at each step exactly one basic variable is replaced, we are actually moving between adjacent vertices. See Problem 10 for a development of these ideas.

By using the corollary of the previous section in the proof of the above corollary, we have made use of the central theorems of this chapter, theorems that have been proved algebraically. In fact, the above result can also be proved independently using only the theory of convex sets. (See, for example, Problem 11.) This suggests an alternative, theoretically sound approach to the linear optimization problem. First, compute all the basic feasible solutions to the problem; second, compare the value of the objective function at each of these points. As long as we know that the function has an optimal value, it must be the optimal value in this set. However, this technique is far from practical; if the constraint system has m equations and n unknowns, there could be up to $\binom{n}{m}$ basic feasible solutions, where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

is the binomial coefficient. For example,

$$\binom{15}{5} = 3003 \text{ and } \binom{20}{10} = 184,756$$

Problem Set 3.9

1. Suppose P and Q are points in $\mathbb{R}^n$. Show geometrically that the set $tP + (1-t)Q = Q + t(P-Q)$, $0 \leq t \leq 1$, is the line segment connecting P and Q.
2. Prove Theorem 3.9.1. (*Hint.* Use Example 3.9.2 as a model.)
3. Prove Theorem 3.9.2.
4. Prove Corollary 3.9.1.
5. Theorems 3.9.1 and 3.9.2 imply immediately that a hyperplane is convex. Why?
6. True or false:
   (a) The union of two convex sets is convex.
   (b) The complement of a convex set is convex.
7. True or false: A point P is a vertex of a convex set S if and only if P is not the interior point of any line segment in S. (An interior point of a line segment L is any point of L other than the two end points.)
8. Prove that if P and Q are vertices of a convex set S and $X = P + t(Q-P)$ is a point of S, then $0 \leq t \leq 1$.
9. Consider the general linear programming problem (3.4.1) on page 78. Suppose $P = (b_1, \ldots, b_m, 0, \ldots, 0)$ and $Q = (0, b_2^*, \ldots, b_m^*, b_{m+1}^*, \ldots, 0)$ are distinct basic

CHAPTER 4

DUALITY

4.1 INTRODUCTION TO DUALITY

Frequently in mathematics there exist relationships between concepts, systems, or problems that are not immediately apparent but, once understood, reap many dividends. For example, consider in calculus the relationship between the integral and the derivative expressed in the Fundamental Theorem of Calculus, or in linear algebra, the relationship between linear transformations and matrices. Relationships such as these not only can be used for practical or computational purposes, but also can provide a unified and coherent theory, so that insights and techniques from one system can contribute to the understanding and usefulness of another.

In this chapter we will develop one such unifying notion, the concept of *duality*. For any linear programming problem, the associated dual linear programming problem will be defined. In Section 4.3 it will be shown that in certain optimization situations, the dual problem arises quite naturally; and in Sections 4.4 and 4.5 important theoretical results relating the two problems will be developed. In particular, in Section 4.4 the fundamental Duality Theorem will be proved.

The concept of duality plays an important role in the remainder of the text. In Section 5.1, we will expand upon the ideas in Sections 4.3 and 4.4 to yield a sensitivity analysis procedure useful in a variety of applications. In Section 5.6 the Dual Simplex Algorithm will be developed, and in Section 7.2 the Transportation Problem Algorithm, a primal-dual algorithm, will be developed. Later, in Chapter 9, when we consider two-person zero-sum games, we will see that the problem of solving such a game is equivalent to solving a linear programming problem and its dual problem, and that the question of the existence of a solution to these games is answered using the Duality Theorem.

We conclude this section with an example that should provide some motivation for the definitions to follow in Section 4.2.

Example 4.1.1. To obtain favorable bulk rates, a soft ice cream producer negotiates 6-month contracts in early summer with distant wholesalers for the weekly purchase of fixed quantities of cream, skim milk, and chocolate syrup. However, in the fall, when the demand for soft ice cream decreases, the producer will be left with a surplus of these three quantities. In particular, suppose that in the fall there is weekly 100 gal of cream unused in the production of the ice cream, 300 gal of skim milk, and 60 lb of chocolate syrup.

An Introduction to Linear Programming and Game Theory, Third Edition. By P. R. Thie and G. E. Keough.

To utilize this surplus, the producer bottles and delivers cases of whole and chocolate milk to a local school. A case of whole milk uses 1 gal of cream and 2 gal of skim milk and yields a net gain of \$3 (selling price less bottling and delivery costs); a case of chocolate milk uses 0.4 gal of cream, 2.5 gal of skim milk, and 0.6 lb of chocolate syrup and yields a gain of \$4. Hoping to maximize the net gain attainable with this surplus, the producer formulates the following linear programming problem, with x_1 the number of cases of whole milk and x_2 the number of cases of chocolate milk to be produced each week.

$$\begin{aligned} &\text{Maximize } 3x_1 + 4x_2 \\ &\text{subject to} \\ &x_1 + 0.4x_2 \le 100 \\ &2x_1 + 2.5x_2 \le 300 \\ &\phantom{2x_1 + {}} 0.6x_2 \le 60 \\ &x_1, x_2 \ge 0 \end{aligned} \tag{4.1.1}$$

However, before this problem is solved and contracts are signed with the local school, the producer is contacted by the manager of the town dairy. The dairy also supplies milk to the local school system and, in fact, strives to be the sole such supplier. This would increase the dairy's presence in the town and would also allow the dairy some freedom in negotiating prices for the school contract. To accomplish this, the manager of the dairy offers to simply buy from the ice cream producer his surplus milk and syrup, which the dairy would then use in its own bottling plant.

The offer intrigues the ice cream producer. It would allow him to focus his company on the making and selling of ice cream and, if the dairy's offer is financially sound, to continue the economical bulk rate contracts with the distant wholesalers. But what prices for the surplus ingredients are financially sound to the producer?

To attempt to answer this question, the dairy manager notes that the only value to the producer that the surplus milk and syrup have is in bottling and selling cases of whole milk and chocolate milk. In particular, suppose the manager offers the producer y_1 dollars for each gallon of surplus cream, y_2 dollars for each gallon of skim, and y_3 dollars for each pound of chocolate syrup. Then, since the bottling and delivery of a case of whole milk requires 1 gal of cream and 2 gal of skim milk and yields a gain of \$3, the dairy manager realizes that to be competitive, y_1 and y_2 must be set so that $y_1 + 2y_2 \ge 3$. Similarly, consideration of the input and gain associated with a case of chocolate milk yields the inequality $0.4y_1 + 2.5y_2 + 0.6y_3 \ge 4$. Of course, the dairy manager also wants to keep her total costs down and so, in determining these prices, she is led to the following linear programming problem:

$$\begin{aligned} &\text{Minimize } 100y_1 + 300y_2 + 60y_3 \\ &\text{subject to} \\ &y_1 + 2y_2 \ge 3 \\ &0.4y_1 + 2.5y_2 + 0.6y_3 \ge 4 \\ &y_1, y_2, y_3 \ge 0 \end{aligned} \tag{4.1.2}$$

The linear programming problem (4.1.2) is the dual of the problem (4.1.1). We have been led to these problems by considering the disposal of surplus goods from two different but related perspectives. Other examples in which the dual arises quite naturally will be discussed in Section 4.3. For the time being, let us note some relationships between the two problems (4.1.1) and (4.1.2). (As we will see, these relationships constitute the definition of the dual linear programming problem.)

1. Problem (4.1.2) is a minimization problem with ($\geq$) constraints; (4.1.1) is a maximization problem with ($\leq$) constraints.
2. The number of nonnegative variables in (4.1.2) equals the number of constraints in (4.1.1). (A price was to be set using (4.1.2) for each limited resource in (4.1.1).)
3. The number of constraints in (4.1.2) equals the number of nonnegative variables in (4.1.1). (The y_1, y_2, y_3 had to compare favorably with each of the two processes of (4.1.1).)
4. (a) The coefficients of the objective function of (4.1.2) are the constant terms of the constraints of (4.1.1).
    (b) The constant terms of the constraints of (4.1.2) are the coefficients of the objective function of (4.1.1).
    (c) The coefficients of the constraints of (4.1.2) are the coefficients of the constraints of (4.1.1), with the rows and columns interchanged (transposed).

Problem Set 4.1

The following problems refer to the example of this section.

1. Solve (4.1.1) graphically. What is the maximum the ice cream producer can earn each week with his surplus?
2. (a) Solve (4.1.2) using the simplex algorithm.
    (b) How much should the dairy manager offer the producer for each gallon of cream? Each gallon of skim? Each pound of syrup?
    (c) What is the total amount the dairy manager would be paying the producer each week? Would he accept the offer?

4.2 DEFINITION OF THE DUAL PROBLEM

The definition of the dual problem will initially be given in terms of a linear programming problem expressed in a special form, called the *max form* of the problem. Problems in another special form, a *min form*, are equally useful. We first define these terms.

Definition 4.2.1. A linear programming problem stated in the following form is said to be in *max form*:

$$\begin{array}{l}
\text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} \\
a_{11}x_1 \;+\; a_{12}x_2 \;+\; \ldots \;+\; a_{1n}x_n \;\leq b_1 \\
a_{21}x_1 \;+\; a_{22}x_2 \;+\; \ldots \;+\; a_{2n}x_n \;\leq b_2 \\
\vdots \\
a_{m1}x_1 \;+\; a_{m2}x_2 \;+\; \ldots \;+\; a_{mn}x_n \;\leq b_m \\
x_1, x_2, \ldots, x_n \;\geq\; 0
\end{array} \tag{4.2.1}$$

Thus the max form of a linear programming problem, called simply the *max problem*, is a maximization problem with nonnegative variables and a system of constraints consisting of only ($\leq$) inequalities. Note that there are no restrictions on the signs of the coefficients a_{ij}, constant terms b_i, and coefficients c_j.

Definition 4.2.2. A linear programming problem stated in the following form is said to be in *min form*:

$$\begin{array}{l}
\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} \\
a_{11}x_1 \;+\; a_{12}x_2 \;+\; \ldots \;+\; a_{1n}x_n \;\geq b_1 \\
a_{21}x_1 \;+\; a_{22}x_2 \;+\; \ldots \;+\; a_{2n}x_n \;\geq b_2 \\
\vdots \\
a_{m1}x_1 \;+\; a_{m2}x_2 \;+\; \ldots \;+\; a_{mn}x_n \;\geq b_m \\
x_1, x_2, \ldots, x_n \;\geq\; 0
\end{array} \tag{4.2.2}$$

The *min problem* is a minimization problem with nonnegative variables and a system of constraints consisting of only ($\geq$) inequalities. Again, no restrictions have been placed on the signs of the a_{ij}, b_i, and c_j.

We now define the dual to the max problem (4.2.1). Then we will build on this definition to extend the definition of duality to an arbitrary linear programming problem. As we will see, both the max problem and the min problem (4.2.2) will play equal roles in the summarizing definitions.

Definition 4.2.3. The *dual* of the max problem (4.2.1) is the following linear programming problem:

$$\begin{array}{l}
\text{Minimize } v = b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{subject to} \\
a_{11}y_1 \;+\; a_{21}y_2 \;+\; \ldots \;+\; a_{m1}y_m \;\geq c_1 \\
a_{12}y_1 \;+\; a_{22}y_2 \;+\; \ldots \;+\; a_{m2}y_m \;\geq c_n \\
\vdots \\
a_{1n}y_1 \;+\; a_{2n}y_2 \;+\; \ldots \;+\; a_{mn}y_m \;\geq c_n \\
y_1, y_2, \ldots, y_n \geq 0
\end{array} \tag{4.2.3}$$

Thus the dual to the max problem (4.2.1) with m ($\leq$) constraints and n nonnegative variables is a minimization problem with m nonnegative variables and n ($\geq$) constraints. For each i, $1 \leq i \leq m$, variable y_i of the dual corresponds to the ith constraint of the max problem. The coefficients of y_i in the ith column of the constraints of (4.2.3) are the coefficients of the ith constraint in (4.2.1). Reciprocally, for each j, $1 \leq j \leq n$, the jth constraint in the dual corresponds to the jth variable x_j in (4.2.1); the coefficients of the variables in the jth constraint in the dual are the coefficients of x_j in the constraints of (4.2.1). Note also the interchange between the constant terms of the constraints and the coefficients of the objective functions for the two problems. (Compare the above with the list of relationships given at the end of the example of the previous section.)

Example 4.2.1. The linear programming problem of

$$\begin{aligned}
&\text{Maximizing } 6x_1 + x_2 + 4x_3 \\
&\text{subject to} \\
&3x_1 + 7x_2 + x_3 \leq 15 \\
&x_1 - 2x_2 + 3x_3 \leq 20 \\
&x_1, x_2, x_3 \geq 0
\end{aligned} \tag{4.2.4}$$

has as its dual the problem of

$$\begin{aligned}
&\text{Minimizing } 15y_1 + 20y_2 \\
&\text{subject to} \\
&3y_1 + y_2 \geq 6 \\
&7y_1 - 2y_2 \geq 1 \\
&y_1 + 3y_2 \geq 4 \\
&y_1, y_2 \geq 0
\end{aligned} \tag{4.2.5}$$

Matrix notation can be used to express any linear programming problem and, in particular, the max problem and its dual problem, succinctly. Using (4.2.1), we will define the coefficient matrix A and column vectors b, c, and X as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let A^t denote the transpose of matrix A, and let $c \cdot X$ denote the dot or scalar product of the vectors c and X. Then

$$A^t = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

and

$$c \cdot X = c_1x_1 + c_2x_2 + \cdots + c_nx_n = c^tX = X^tc = X \cdot c$$

The max problem of (4.2.1) is simply to maximize $z = c \cdot X$ subject to $AX \leq b, X \geq 0$, where $AX \leq b$ means that each component of the column vector AX is less than or equal to the corresponding component of the vector b, and $X \geq 0$ is defined similarly, with 0 in this case being the n-dimensional zero vector. Let Y be the m-dimensional column vector $(y_1, y_2, \ldots, y_m)^t$. Then the problem of (4.2.3) is to minimize $v = b \cdot Y$ subject to $A^tY \geq c, Y \geq 0$.

In summary, we have the following:

$$\begin{array}{ll} \text{Max problem:} & \text{Maximize } z = c \cdot X \text{ subject to } AX \leq b, X \geq 0 \\ \text{Dual problem:} & \text{Minimize } v = b \cdot Y \text{ subject to } A^tY \geq c, Y \geq 0 \end{array} \qquad (4.2.6)$$

To extend the definition of duality to an arbitrary problem, first note that any linear programming problem is equivalent to a problem stated in max form. For example, we have already seen how a minimization problem can be transformed into an equivalent maximization problem and unrestricted variables replaced by variables restricted in sign. A constraint involving an equality can be replaced by two inequalities in opposite directions. For example, the set of points $(x_1, x_2) \in \mathbb{R}^2$ such that $3x_1 + 2x_2 = 5$ equals the set of (x_1, x_2) such that $3x_1 + 2x_2 \geq 5$ and $3x_1 + 2x_2 \leq 5$. Finally, the direction of an inequality can be changed by multiplication by (-1).

With this, the dual to any linear programming problem can be constructed. To determine this dual, first express the given problem as an equivalent linear programming problem in max form and then use the above definition.

As an application, let us determine the dual to the min problem of (4.2.3), the dual of (4.2.1). The problem as stated is to minimize $b \cdot Y$ subject to $A^tY \geq c$, $Y \geq 0$. Letting $-M$ denote the matrix found by multiplying all the entries of a matrix M by (-1), the problem of (4.2.3) is equivalent to the problem of

$$\text{Maximizing } (-b) \cdot Y \text{ subject to } (-A^t)Y \leq -c,\ Y \geq 0$$

But this problem is in max form, and its dual is, using (4.2.6), to

$$\text{Minimize } (-c) \cdot X \text{ subject to } (-A^t)^tX \geq -b,\ X \geq 0$$

Using the fact that for any matrix M, $(M^t)^t = M$, this problem is equivalent to the problem of

$$\text{Maximizing } c \cdot X \text{ subject to } AX \leq b,\ X \geq 0$$

Note that this is precisely the problem of (4.2.1). We have proven that the dual of the min problem is a max problem and that for any linear programming problem, the dual of the dual is the original problem. Hence, repeated application of this operation of constructing the dual problem to a given problem does not lead to a chain of distinct problems but, instead, cycles after two steps, resulting in exactly two problems, each the dual of the other.

Example 4.2.2. The linear programming problem of

$$\begin{aligned}
&\text{Minimizing } 12x_1 + 9x_2 - 2x_3\\
&\text{subject to}\\
&8x_1 + 3x_2 + 5x_3 \geq 6\\
&x_1 \qquad\quad - 3x_3 \geq -4\\
&x_1, x_2, x_3 \geq 0
\end{aligned}$$

is in min form, and thus, from the above, we can write immediately that its dual is to

$$\begin{aligned}
&\text{Maximize } 6y_1 - 4y_2\\
&\text{subject to}\\
&8y_1 + y_2 \leq 12\\
&3y_1 \qquad \leq 9\\
&5y_1 - 3y_2 \leq -2\\
&y_1, y_2 \geq 0
\end{aligned}$$

We consider now the steps involved in the construction of the dual of a problem first, having an equality constraint, and second, having an unrestricted variable.

Example 4.2.3. To determine the dual of the problem of

$$\begin{aligned}
&\text{Maximizing } 6x_1 + x_2 + 4x_3\\
&\text{subject to}\\
&3x_1 + 7x_2 + x_3 \leq 15\\
&x_1 - 2x_2 + 3x_3 = 20\\
&x_1, x_2, x_3 \geq 0
\end{aligned} \tag{4.2.7}$$

notice that this is the problem of Example 4.2.1 with the second constraint changed to an equality. We replace the equality constraint by two inequalities and multiply the resulting $(\geq)$ inequality by (-1) to find the equivalent problem in max form of

$$\begin{aligned}
&\text{Maximizing } 6x_1 + x_2 + 4x_3\\
&\text{subject to}\\
&3x_1 + 7x_2 + x_3 \leq 15\\
&x_1 - 2x_2 + 3x_3 \leq 20\\
&-x_1 + 2x_2 - 3x_3 \leq -20\\
&x_1, x_2, x_3 \geq 0
\end{aligned} \tag{4.2.8}$$

Using variables y_1, y_2, y_3, the dual to (4.2.8) is to

$$\begin{aligned}
&\text{Minimize } 15y_1 + 20y_2 - 20y_3\\
&\text{subject to}\\
&3y_1 + y_2 - y_3 \geq 6\\
&7y_1 - 2y_2 + 2y_3 \geq 1\\
&y_1 + 3y_2 - 3y_3 \geq 4\\
&y_1, y_2, y_3 \geq 0
\end{aligned} \tag{4.2.9}$$

which can be rewritten as

$$\begin{aligned}
&\text{Minimize } 15y_1 + 20(y_2 - y_3) \\
&\text{subject to} \\
&3y_1 + \ \ (y_2 - y_3) \geq 6 \\
&7y_1 - 2(y_2 - y_3) \geq 1 \\
&\ \ y_1 + 3(y_2 - y_3) \geq 4 \\
&y_1, y_2, y_3 \geq 0
\end{aligned} \tag{4.2.10}$$

which is equivalent to

$$\begin{aligned}
&\text{Minimize } 15y_1 + 20y_4 \\
&\text{subject to} \\
&3y_1 + \ \ y_4 \geq 6 \\
&7y_1 - 2y_4 \geq 1 \\
&\ \ y_1 + 3y_4 \geq 4 \\
&y_1 \geq 0, y_4 \text{ unrestricted}
\end{aligned} \tag{4.2.11}$$

Note that (4.2.11), the dual to (4.2.7), is almost (4.2.5), the dual to (4.2.4). The difference is that in (4.2.11), the variable y_4 corresponding to the equality constraint in (4.2.7) is unrestricted. Clearly, the algebra above generalizes. When defining a dual, any variable in the dual corresponding to an equality constraint in the original problem is unrestricted in sign.

Example 4.2.4. To determine the dual of (4.2.11), a problem in min form except for an unrestricted variable, we first replace the unrestricted variable with the difference of two nonnegative variables (4.2.10), simplify to a problem in min form (4.2.9), write the dual (4.2.8), and replace the last two inequalities with the equivalent equality. This yields (4.2.7), the dual to (4.2.11); and the constraint in the dual generated by the unrestricted variable y_4 in the original problem is an equality. Again, we can generalize. Constraints in a dual corresponding to unrestricted variables in the original problem are equality constraints.

Combining these observations, we summarize the construction of the dual to an arbitrary linear programming problem. First, express the problem, using nonnegative and unrestricted variables, as either a maximization problem with ($\leq$) and equality constraints or a minimization problem with ($\geq$) and equality constraints. The dual can then be immediately formulated.

The dual to a maximization problem is a minimization problem with ($\geq$) and equality constraints, and the dual to a minimization problem is a maximization problem with ($\leq$) and equality constraints. In both cases, unrestricted variables in the original problem generate equality constraints in the associated dual; and reciprocally, equality constraints in the original generate unrestricted variables in the dual problem. Table 4.1 summarizes the relationships.

Table 4.1

Max Problem	*← dual →*	*Min Problem*
ith ($\leq$) inequality		ith nonnegative variable
ith (=) constraint		ith unrestricted variable
jth nonnegative variable		jth ($\geq$) inequality
jth unrestricted variable		jth (=) constraint
Objective function coefficients		Constant terms of constraints
Constant terms of constraints		Objective function coefficients
Coefficient matrix of constraints A		Coefficient matrix of constraints A^t

Example 4.2.5. The linear programming problem of

$$\begin{array}{l} \text{Minimizing } x_1 - 2x_2 + 3x_3 \\ \text{subject to} \\ 4x_1 + 5x_2 - 6x_3 = 7 \\ 8x_1 - 9x_2 + 10x_3 \leq 11 \\ x_1, x_2 \geq 0, x_3 \text{ unrestricted} \end{array}$$

is equivalent to the problem of

$$\begin{array}{l} \text{Minimizing } x_1 - 2x_2 + 3x_3 \\ \text{subject to} \\ 4x_1 + 5x_2 - 6x_3 = 7 \\ -8x_1 + 9x_2 - 10x_3 \geq -11 \\ x_1, x_2 \geq 0, x_3 \text{ unrestricted} \end{array}$$

and therefore has as its dual the problem of

$$\begin{array}{l} \text{Maximizing } 7y_1 - 11y_2 \\ \text{subject to} \\ 4y_1 - 8y_2 \leq 1 \\ 5y_1 + 9y_2 \leq -2 \\ -6y_1 - 10y_2 = 3 \\ y_1 \text{ unrestricted}, y_2 \geq 0 \end{array}$$

Example 4.2.6. The linear programming problem of

$$\begin{array}{l} \text{Maximizing } 12x_1 + 2x_2 \\ \text{subject to} \\ 8x_1 - x_2 \leq 21 \\ x_1 - 6x_2 \geq 13 \\ 3x_1 - 4x_2 = -11 \\ x_1 \text{ unrestricted}, x_2 \geq 0 \end{array}$$

is equivalent to the problem of

$$\begin{aligned}&\text{Maximizing } 12x_1 + 2x_2\\ &\text{subject to}\\ &8x_1 - x_2 \leq 21\\ &-x_1 + 6x_2 \leq -13\\ &3x_1 - 4x_2 = -11\\ &x_1 \text{ unrestricted}, x_2 \geq 0\end{aligned}$$

and therefore has as its dual the problem of

$$\begin{aligned}&\text{Minimizing } 21y_1 - 13y_2 - 11y_3\\ &\text{subject to}\\ &8y_1 - y_2 + 3y_3 = 12\\ &-y_1 + 6y_2 - 4y_3 \geq 2\\ &y_1, y_2 \geq 0, y_3 \text{ unrestricted}\end{aligned}$$

Problem Set 4.2

1. Determine the dual of each of the following linear programming problems.

(a) Maximize $20x_1 + 30x_2$
subject to
$$\begin{aligned}5x_1 - 4x_2 &\leq 100\\ -x_1 + 12x_2 &\leq 90\\ x_2 &\leq 500\\ x_1, x_2 &\geq 0\end{aligned}$$

(b) Minimize $4x_1 - 3x_2$
subject to
$$\begin{aligned}6x_1 + 11x_2 &\geq -30\\ 2x_1 - 7x_2 &\leq 50\\ x_2 &\leq 80\\ x_1, x_2 &\geq 0\end{aligned}$$

(c) Maximize $-x_1 + 2x_2$
subject to
$$\begin{aligned}5x_1 + x_2 &\leq 60\\ 3x_1 - 8x_2 &\geq 10\\ x_1 + 7x_2 &= 20\\ x_1, x_2 &\geq 0\end{aligned}$$

(d) Minimize $6x_1 + 12x_2 - 18x_3$
subject to

$$\begin{aligned} x_1 - 3x_2 + 6x_3 &= 30 \\ 2x_1 + 8x_2 - 16x_3 &= 70 \\ x_1, x_2 \geq 0, x_3 &\text{ unrestricted} \end{aligned}$$

(e) Maximize $x_1 - 7x_2 + 3x_3$
subject to

$$\begin{aligned} 2x_2 + 5x_3 &= 20 \\ 8x_1 - 3x_3 &= 40 \\ x_2 + 4x_3 &\geq 60 \\ x_1, x_3 \geq 0, x_2 &\text{ unrestricted} \end{aligned}$$

(f) Minimize $2y_1 - 3y_2 + 4y_3$
subject to

$$\begin{aligned} 8y_1 - y_3 &= 50 \\ 6y_2 + y_3 &\leq 60 \\ y_1, y_2 \geq 0, -15 \leq y_3 &\leq 0 \end{aligned}$$

2. (a) Determine the dual to the problem of
Maximizing $x_1 - 2x_2$
subject to

$$\begin{aligned} x_2 &\geq 1 \\ x_1 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b) Rewrite your answer to part (a) as an equivalent maximization problem.

(c) Compare your response in part (b) to the original problem of part (a). Observation?

(d) Show that the following problem is also its own dual.
Maximizing $x_1 - 2x_2 - 3x_3$
subject to

$$\begin{aligned} x_2 + 2x_3 &\geq 1 \\ x_1 + 3x_3 &\leq 2 \\ 2x_1 - 3x_2 &= 3 \\ x_1, x_2 \geq 0, x_3 &\text{ unrestricted} \end{aligned}$$

3. Consider the linear programming problem of Example 4.2.1 of this section.

(a) Show that the objective function of the dual problem is bounded below.

(b) Solve the dual problem graphically.

(c) Solve the maximization problem using the simplex method. Note that the optimal values of the objective functions are equal.

(d) Compare the bottom two entries in the slack variable columns of the last simplex tableau of part (c) with the point in part (b) that yielded the minimal value.

fruit, using two prices: one for a bushel of choice produce and the other for regular produce. Considering that the student must convince the grower that it is to his advantage to let her supervise the harvest, how should she set these three costs?

5. Consider Problem 11 of Section 2.3.
   (a) Formulate the associated linear programming problem.
   (b) Determine the dual problem.
   (c) Suppose the manager of the electronics firm wants to assess the value of a unit of material and a unit of labor in the production and sale of the circuits. To do this, she lets $\$y_1$ and $\$y_2$ denote these two values. The circuit for a radio requires 2 units of material and 1 unit of labor and sells for \$8. The manager reasons, therefore, that 2 units of material plus 1 unit of labor must be worth at least \$8, but could be worth more if these units can be used in the production of other types of circuits that are more profitable. Thus she sets $2y_1+y_2 \geq 8$. The manager continues in this manner. Compare the resulting problem with the problem determined in part (b). (Note that the Duality Theorem guarantees that the optimal values for the problems of parts (a) and (b) are equal.)

4.4 THE DUALITY THEOREM

In this section we prove the celebrated Duality Theorem. It is generally accepted that John von Neumann was the first mathematician to recognize the significance of the duality principle in this setting and endeavor to develop a proof of the Duality Theorem.

We start with the max problem of (4.2.1), the problem of maximizing $z = c\cdot X$ subject to $AX \leq b$, $X \geq 0$. The dual to this problem is to minimize $v = b\cdot Y$ subject to $AY \geq c$, $Y \geq 0$. We will show first that the set of possible values for the objective function z of the max problem lies to the left of the set of possible values for the function v. Then, with this result, we will prove the Duality Theorem using the simplex method and, in particular, Theorem 3.8.1.

Theorem 4.4.1. *Suppose X_0 is a feasible solution to the problem of maximizing $c\cdot X$ subject to $AX \leq b$, $X \geq 0$ and Y_0 is a feasible solution to the dual problem of minimizing $b\cdot Y$ subject to $AY \geq c$, $Y \geq 0$. Then*

$$c\cdot X_0 \leq b\cdot Y_0$$

Proof. Since X_0 is a feasible solution to the max problem with constraints $AX \leq b$, where A is an $m\times n$ matrix, the $m\times 1$ vector $u = b - AX_0 \geq 0$. In fact, the m components of u are the slack in the m inequalities of $AX_0 \leq b$. Similarly, Y_0 a feasible solution to the dual implies that $A^tY_0 \geq c$, and so the column vector $v = A^tY_0 - c$ of slack in this set of n inequalities also has nonnegative components. Using these vectors, we can write

$$AX_0 = b - u \quad \text{and} \quad A^tY_0 = c + v$$

Now since the product $\underbrace{Y_0^t}_{1\times m}\underbrace{A}_{m\times n}\underbrace{X_0}_{n\times 1}$ is a real number, we have $Y_0^tAX_0 = (Y_0^tAX_0)^t = X_0^tA^tY_0$, and so

$$Y_0^tAX_0 = Y_0^t(AX_0) = Y_0^t(b-u) \text{ equals } X_0^tA^tY_0 = X_0^t(A^tY_0) = X_0^t(c+v)$$

that is,

$$Y_0^tb - Y_0^tu = X_0^tc + X_0^tv$$

Thus, since $u, v, X_0, Y_0 \geq 0$,

$$b\cdot Y_0 - c\cdot X_0 = \underbrace{u\cdot Y_0}_{\geq 0} + \underbrace{v\cdot X_0}_{\geq 0} \geq 0 \qquad \square$$

We state the first corollary below for future reference in Section 4.5. The two subsequent corollaries are for immediate use in this section.

Corollary 4.4.1. *If X_0 is a feasible solution to the problem of maximizing $c\cdot X$ subject to $AX \leq b$, $X \geq 0$ and Y_0 is a feasible solution to the problem of minimizing $A^tY \geq c$, $Y \geq 0$, then*

$$b\cdot Y_0 - c\cdot X_0 = (b - AX_0)\cdot Y_0 + (A^tY_0 - c)\cdot X_0$$

Proof. This is the equality statement of the last line of the above proof. □

Corollary 4.4.2. *If X_0 and Y_0 are feasible solutions to the max and min problems, respectively, and if $c\cdot X_0 = b\cdot Y_0$, then the optimal values of the objective functions z and v equal this common value; that is, maximum $z = c\cdot X_0 = b\cdot Y_0 =$ minimum v and X_0 and Y_0 are optimal solution points for their respective problems.*

Proof. Suppose X_1 is any feasible solution to the max problem. Then, from the theorem, $c\cdot X_1 \leq b\cdot Y_0$, so $c\cdot X_1 \leq c\cdot X_0$. Thus the maximum value of the function $z = c\cdot X$ is $c\cdot X_0$. Similarly for the dual problem. □

Corollary 4.4.3. *If the objective function z of the max problem is not bounded above, the min problem has no feasible solutions. Similarly, if the objective function v of the min problem is not bounded below, the max problem has no feasible solutions.*

The proof of Corollary 4.4.3 is left to the reader (Problem 1). The converse to this corollary is false. Examples can be constructed for which neither the max problem nor its dual, the min problem, have feasible solutions (see Problem 2).

Theorem 4.4.2 (Duality Theorem). *Suppose either the problem of*

$$\textit{Maximizing } z = c\cdot X \textit{ subject to } AX \leq b,\ X \geq 0$$

or the problem of

$$\textit{Minimizing } v = b\cdot Y \textit{ subject to } A^tY \geq c,\ Y \geq 0$$

has a finite optimal solution. Then so does the other problem, and the optimal values of the objective functions are equal, that is,

$$\text{Max}\, z = \text{Min}\, v$$

Proof. Assume first that the max problem has a finite optimal solution. Thus we assume the existence of an X_0 such that $AX_0 \leq b$, $X_0 \geq 0$ and, for any other X with $AX \leq b, X \geq 0$, we have $c \cdot X \leq c \cdot X_0$.

Now the solution to the min problem will be found by applying the simplex method to the max problem. To do this, we first write the max problem in standard form by adding m slack variables x_j, $n+1 \leq j \leq n+m$, and multiply the objective function by -1. This gives the problem of

$$\begin{array}{l}
\text{Minimizing } -c_1x_1 - c_2x_2 - \cdots - c_nx_n = -z \\
\text{subject to} \\
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} \qquad\qquad\qquad = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad + x_{n+2} \qquad\qquad = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \qquad\qquad\qquad + x_{m+n} = b_m \\
x_j \geq 0, 1 \leq j \leq n+m
\end{array} \tag{4.4.1}$$

We now assume in our proof that the constants b_i, $1 \leq i \leq m$, are nonnegative. If this is the case, the above problem is in canonical form with basic variables x_{n+1}, x_{n+2}, $\ldots, x_{n+m}$, since the associated basic solution is feasible, and the simplex method can be initiated directly commencing with the second stage.

(Recall that in Section 4.2 when the max and min problems were defined, no restrictions were placed on the constants. Thus, with this assumption, our proof loses some generality. The extension of the proof to the general case is developed in Problem 8.)

From Theorem 3.8.1, we know that there is a finite sequence of pivot operations driving the problem of (4.4.1) to the optimal value of the objective function. The initial tableaux for such a sequence would have a form such as

	x_1	x_2	$\ldots$	x_n	x_{n+1}	$\ldots$	x_{n+m}	
x_{n+1}	a_{11}	a_{12}	$\ldots$	a_{1n}	1	0	$\ldots$	b_1
x_{n+2}	a_{21}	a_{22}	$\ldots$	a_{2n}	0	1	$\ldots$	b_2
$\vdots$								
x_{n+m}	a_{m1}	a_{m2}	$\ldots$	a_{mn}	0	$\ldots$	1	b_m
	$-c_1$	$-c_2$	$\ldots$	$-c_n$	0	0	0	0

and the final tableau would assume the form

	x_1	x_2	$\ldots$	x_n	x_{n+1}	$\ldots$	x_{n+m}	
	r_1	r_2	$\ldots$	r_n	s_1	$\ldots$	s_m	$c \cdot X_0$

Since our concern will be with only the bottom row of this last tableau, we have used the symbols r_j, $1 \leq j \leq n$ and s_i, $1 \leq i \leq m$ to denote the numbers appearing in these positions and have left the other positions of the tableau blank. Since this tableau represents the final step of the simplex process in the problem of (4.4.1), we have $r_j \geq 0$ and $s_i \geq 0$ for $1 \leq j \leq n$, $1 \leq i \leq m$, and the minimum of $-z$ is $-c \cdot X_0$.

Let Y_0 be the column vector $(s_1, s_2, \ldots, s_m)^t$. We will show that

(a) $Y_0 \geq 0$
(b) $A^t Y_0 \geq c$
(c) $b \cdot Y_0 = c \cdot X_0$

As has already been mentioned, $Y_0 \geq 0$. To show (b) and (c), consider the equation represented by the bottom row of the final tableau:

$$r_1x_1 + \cdots + r_nx_n + s_1x_{n+1} + \cdots + s_mx_{n+m} = c \cdot X_0 + (-z)$$

This equation represents the result of all the pivot operations on the initial equation for the objective function

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0 + (-z)$$

And, at each pivot step, some linear combination of the original constraining equations was added to this equation for the objective function. Thus there exist m constants, t_i, $1 \leq i \leq m$, such that when the $(m+1)$ equations

$$\begin{array}{llllllll}
t_1(a_{11}x_1 & + & a_{12}x_2 & + \ldots + & a_{1n}x_n & + x_{n+1} & & = b_1) \\
t_2(a_{21}x_1 & + & a_{22}x_2 & + \ldots + & a_{2n}x_n & & + x_{n+2} & = b_2) \\
\vdots & & & & & & & \\
t_m(a_{m1}x_1 & + & a_{m2}x_2 & + \ldots + & a_{mn}x_n & & + x_{m+n} & = b_m) \\
(-c_1x_1 & - & c_2x_2 & - \ldots - & c_nx_n & & & = -z)
\end{array}$$

are added together, the result is the equation

$$r_1x_1 + \cdots + r_nx_n + s_1x_{n+1} + \cdots + s_mx_{n+m} = c \cdot X_0 + (-z)$$

Comparing the coefficients of the slack variables, we see that $s_i = t_i$ for $1 \leq i \leq m$. Using this result and comparing the coefficients of x_1, we have

$$s_1a_{11} + s_2a_{21} + \cdots + s_ma_{m1} - c_1 = r_1 \geq 0$$

and so

$$s_1a_{11} + s_2a_{21} + \cdots + s_ma_{m1} \geq c_1$$

Similarly, comparing the coefficients of x_j for any j, $1 \leq j \leq n$, we have

$$s_1a_{1j} + s_2a_{2j} + \cdots + s_ma_{mj} - c_j = r_j \geq 0$$

and so

$$s_1a_{1j} + s_2a_{2j} + \cdots + s_ma_{mj} \geq c_j$$

Thus

$$A^t Y_0 \geq c$$

To show (c), consider the constant terms in the above equations. We must have

$$s_1 b_1 + s_2 b_2 + \cdots + s_m b_m = c \cdot X_0$$

that is,

$$b \cdot Y_0 = c \cdot X_0$$

Since $Y_0 \geq 0$ and $A^t Y_0 \geq c$, the point Y_0 is a feasible solution to the min problem. The value of the objective function v at Y_0, $b \cdot Y_0$, is equal to the value of the objective function z at X_0. Thus, from Corollary 4.4.2, the minimal value of v is $b \cdot Y_0$, so the optimal values of both problems are equal.

Finally, suppose that we know initially that it is the min problem that has the finite optimal solution. But in Section 4.2 it was shown that this problem is equivalent to a problem expressed in max form. Thus we can apply what we have already proved to this equivalent problem and conclude that the dual to the min problem, the max problem, has the same optimal solution. □

Corollary 4.4.4. *If both the max and min problems have feasible solutions, then both objective functions have optimal solutions and* Max z = Min v.

Proof. Since both problems have feasible solutions, it follows from Theorem 4.4.1 that the objective function z is bounded above and the objective function v is bounded below. From Corollary 3.8.1, both objective functions attain their optimal values and, from the Duality Theorem, these optimal values must be equal. □

In summary, we have shown that there are exactly four different categories into which solutions to the max and min problems can fall.

1. Both problems have feasible solutions. Then the sets of possible values for the objective functions z and v relate on the real line as follows:

$$\underset{\text{optimal value for both}}{\underset{\downarrow}{\underline{\quad z = c \cdot X \qquad | \qquad v = b \cdot Y \quad}}}$$

2. The objective function z is unbounded above and the min problem has no feasible solutions.
3. The objective function v is unbounded below and the max problem has no feasible solutions.
4. Both problems have no feasible solutions.

The following example demonstrates an important application of the duality theorem.

Example 4.4.1. Suppose we apply the simplex algorithm to the problem of

$$\begin{array}{l} \text{Maximizing } -5x_1+18x_2+6x_3-x_4 \\ \text{subject to} \\ \begin{array}{rcrcrcrcl} 2x_1 & & & - & x_3 & + & 3x_4 & \leq & 20 \\ & & x_2 & - & 2x_3 & - & x_4 & \leq & 30 \\ -3x_1 & + & 6x_2 & + & 3x_3 & + & 4x_4 & \leq & 24 \end{array} \\ x_1,x_2,x_3,x_4 \geq 0 \end{array} \tag{4.4.2}$$

and the resulting final tableau suggests a maximum value of 112 for the objective function attained at the point $(10,9,0,0)$ (and an optimal solution point of $(2,0,3)$ for the dual). We can now easily check the accuracy of our calculations.

First, is the point $(10,9,0,0)$ a feasible solution to (4.4.2), and is the value of the objective function at this point 112? (It might be hoped that this part of the test procedure is already standard practice.)

Second, consider the dual to (4.4.2)

$$\begin{array}{l} \text{Minimize } 20y_1+30y_2+24y_3 \\ \text{subject to} \\ \begin{array}{rcrcrcr} 2y_1 & & & - & 3y_3 & \geq & -5 \\ & & y_2 & + & 6y_3 & \geq & 18 \\ -y_1 & - & 2y_2 & + & 3y_3 & \geq & 6 \\ 3y_1 & - & y_2 & + & 4y_3 & \geq & -1 \end{array} \\ y_1,y_2,y_3 \geq 0 \end{array}$$

Now we determine whether the point $(2,0,3)$ is a feasible solution to this problem and whether the value of the associated objective function at $(2,0,3)$ is also 112.

The answers to the above questions are all positive, as the reader may confirm. Corollary 4.4.2 guarantees then that we have calculated accurately and that our proposed optimal value and solution points are correct. The Duality Theorem guarantees that this test procedure is always available.

From the proof of the Duality Theorem, we know that when the simplex algorithm is applied to a maximization problem with $(\leq)$ constraints, the entries in the bottom row of the final tableau in the slack variable columns give the optimal solution point to the corresponding dual minimization problem. (We had already seen an example of this in the tableau solution of the maximization problem of (4.3.2), the dual problem constructed in Example 4.3.1.) The following example exploits this result.

Example 4.4.2. Consider the linear programming problem of

$$\begin{array}{l}\text{Minimizing } 20x_1 + 15x_2 + 54x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 6x_3 & \geq & 30 \\ & & x_2 & + & 2x_3 & \geq & 6 \\ 2x_1 & & & - & 3x_3 & \geq & -5 \\ x_1 & - & x_2 & & & \geq & 18 \end{array} \\ x_1, x_2, x_3 \geq 0 \end{array}$$

To solve this problem using the simplex method, we would first add 4 slack variables, then 3 artificial variables (the slack variable in the third constraint could serve as a basic variable), and use the full two stages of the algorithm on the resulting problem of 4 constraints and 10 variables. However, the dual to this problem is to

$$\begin{array}{l}\text{Maximize } 30y_1 + 6y_2 - 5y_3 + 18y_4 \\ \text{subject to} \\ \begin{array}{rcrcrcrcr} y_1 & & & + & 2y_3 & + & y_4 & \leq & 20 \\ -2y_1 & + & y_2 & & & - & y_4 & \leq & 15 \\ 6y_1 & + & 2y_2 & - & 3y_3 & & & \leq & 54 \end{array} \\ y_1, y_2, y_3, y_4 \geq 0 \end{array}$$

Applying the simplex algorithm to this dual problem is somewhat easier. Adding three slack variables and solving, we have the tableaux of Table 4.3. The maximum value of the objective function $30y_1 + 6y_2 - 5y_3 + 18y_4$ is 522, and therefore the minimum value of the objective function of the original problem also is 522. Moreover, from the bottom row of the final tableau, we see that the point $(18, 0, 3)$ is an optimal solution point to the original problem. (Of course, the application of the simplex algorithm to the dual of the minimization problem is facilitated here by the fact that the coefficients in the original objective function, 20, 15, and 54, are all nonnegative. If this had not been the case, computing the solution to the dual with the simplex algorithm would also have required the use of artificial variables.)

These observations suggest a general question. If we solve any linear programming problem with a finite optimal solution using the simplex algorithm, can we always find in the final tableau an optimal solution point to the dual? We address this issue in the following examples, considering first the resolution of a minimization problem.

Example 4.4.3. Consider the problem of Example 4.3.1 of

$$\begin{array}{l}\text{Minimizing } 10x_1 + 4x_2 \\ \text{subject to} \\ \begin{array}{rcrcr} 3x_1 & + & 2x_2 & \geq & 60 \\ 7x_1 & + & 2x_2 & \geq & 84 \\ 3x_1 & + & 6x_2 & \geq & 72 \end{array} \\ x_1, x_2 \geq 0 \end{array}$$

Table 4.3

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	
y_5	1	0	2	1	1	0	0	20
y_6	-2	1	0	-1	0	1	0	15
y_7	$\textcircled{6}$	2	-3	0	0	0	1	54
	-30	-6	5	-18	0	0	0	0
y_5	0	$-\frac{1}{3}$	$\frac{5}{2}$	$\textcircled{1}$	1	0	$-\frac{1}{6}$	11
y_6	0	$\frac{5}{3}$	-1	-1	0	1	$\frac{1}{3}$	33
y_1	1	$\frac{1}{3}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{6}$	9
	0	4	-10	-18	0	0	5	270
y_4	0	$-\frac{1}{3}$	$\frac{5}{2}$	1	1	0	$-\frac{1}{6}$	11
y_6	0	$\frac{4}{3}$	$\frac{3}{2}$	0	1	1	$\frac{1}{6}$	44
y_1	1	$\textcircled{\frac{1}{3}}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{6}$	9
	0	-2	35	0	18	0	2	468
y_4	1	0	2	1	1	0	0	20
y_6	-4	0	$\frac{7}{2}$	0	1	1	$-\frac{1}{2}$	8
y_2	3	1	$-\frac{3}{2}$	0	0	0	$\frac{1}{2}$	27
	6	0	32	0	18	0	3	522

Table 4.4

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_6	3	2	-1	0	0	1	0	0	60
x_7	7	2	0	-1	0	0	1	0	84
x_8	3	6	0	0	-1	0	0	1	72
	10	4	0	0	0	0	0	0	0
	-13	-10	1	1	1	0	0	0	-216
x_5	0	0	$-\frac{9}{2}$	$\frac{3}{2}$	1	$\frac{9}{2}$	$-\frac{3}{2}$	-1	72
x_1	1	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	6
x_2	0	1	$-\frac{7}{8}$	$\frac{3}{8}$	0	$\frac{7}{8}$	$-\frac{3}{8}$	0	21
	0	0	1	1	0	-1	-1	0	-144

Subtracting three slack variables, adding three artificial variables, and then using the simplex algorithm yields the initial and final tableaux of Table 4.4.

We know from Example 4.3.1 that the optimal solution point for the corresponding dual maximization problem is $y_1 = 1$, $y_2 = 1$, $y_3 = 0$. Note that these values are precisely the numbers in the bottom row of the final tableau in the slack variable columns for the first, second, and third constraints, respectively.

This is always the case when starting with a minimization problem with ($\geq$) constraints: a solution point to the dual is given in the bottom row of the final tableau in the slack variable columns. A proof of this fact is called for in Problem 11. The proof essentially duplicates the proof in the Duality Theorem, with some minor adjustments (here, for example, $s_j = -t_j$, $1 \leq j \leq m$).

These results can be generalized. In a final tableau presenting the optimal value and an optimal solution point for a linear programming problem, the values of the variables in an optimal solution point to the dual for those variables that correspond to inequalities in the original problem are found in the bottom row of the final tableau in the associated slack variable columns.

Example 4.4.4. The dual to the problem of

$$\begin{array}{l} \text{Maximizing } 3x_1 + x_2 - x_3 \\ \text{subject to} \\ \begin{array}{rrrrl} x_1 + & x_2 + & 5x_3 + & x_4 & \leq 200 \\ -x_1 & & + 2x_3 & & \geq \ 20 \\ & 2x_2 - & x_3 + & 5x_4 & \geq \ 50 \end{array} \\ x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

is the problem of

$$\begin{array}{l} \text{Minimizing } 200y_1 - 20y_2 - 50y_3 \\ \text{subject to} \\ \begin{array}{rrrl} y_1 + & y_2 & & \geq \ 3 \\ y_1 & & - 2y_3 & \geq \ 1 \\ 5y_1 - & 2y_2 + & y_3 & \geq -1 \\ y_1 & & - 5y_3 & \geq \ 0 \end{array} \\ y_1, y_2, y_3 \geq 0 \end{array}$$

The initial and final tableau resolution of the maximization problem is in Table 4.5.

The dual variables y_1, y_2, and y_3 correspond to the first, second, and third inequalities, with slack variables x_5, x_6, and x_7, respectively, of the original problem. Thus an optimal solution point to the dual is $y_1 = 1$, $y_2 = 3$, $y_3 = 0$. This is easy to verify. Note that the point $(1,3,0)$ satisfies the dual constraints and has the required optimal value of 140 at the objective function.

The last two examples in this section contain equality constraints in the original problem and thus unrestricted variables in the dual.

Example 4.4.5. The problem of

$$\begin{array}{l} \text{Maximizing } 3x_1 + 5x_2 + 9x_3 \\ \text{subject to} \\ \begin{array}{rrrl} 4x_1 + & 12x_2 + & 15x_3 & = 900 \\ -x_1 + & 2x_2 + & 3x_3 & = 120 \end{array} \\ x_1, x_2, x_3 \geq 0 \end{array}$$

Table 4.5

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	
x_5	1	1	5	1	1	0	0	0	0	200
x_8	-1	0	2	0	0	-1	0	1	0	20
x_9	0	2	-1	5	0	0	-1	0	1	50
	-3	-1	1	0	0	0	0	0	0	0
	1	-2	-1	-5	0	1	1	0	0	-70
x_7	$\frac{15}{2}$	0	0	-3	2	$\frac{11}{2}$	1	$-\frac{11}{2}$	-1	240
x_3	$-\frac{1}{2}$	0	1	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	10
x_2	$\frac{7}{2}$	1	0	1	1	$\frac{5}{2}$	0	$-\frac{5}{2}$	0	150
	1	0	0	1	1	3	0	-3	0	140

Table 4.6

	x_1	x_2	x_3	x_4	x_5	
x_4	4	12	15	1	0	900
x_5	-1	2	3	0	1	120
	-3	-5	-9	0	0	0
	-3	-14	-18	0	0	-1020
x_1	1	$\frac{2}{9}$	0	$\frac{1}{9}$	$-\frac{5}{9}$	$\frac{100}{3}$
x_3	0	$\frac{20}{27}$	1	$\frac{1}{27}$	$\frac{4}{27}$	$\frac{460}{9}$
	0	$\frac{7}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	560

with the dual problem of

$$\begin{aligned}
&\text{Minimizing } 900y_1 + 120y_2 \\
&\text{subject to} \\
&4y_1 - \ \ y_2 \geq 3 \\
&12y_1 + 2y_2 \geq 5 \\
&15y_1 + 3y_2 \geq 9 \\
&y_1, y_2 \text{ unrestricted}
\end{aligned}$$

has a maximum value of 560 and an optimal solution point of $(\frac{100}{3}, 0, \frac{400}{9})$, as seen in what we'll refer to as the *reduced tableaux resolution* of the problem in Table 4.6, where only the first and last tableaux are displayed. The unrestricted variables y_1 and y_2 of the dual correspond to the two equalities in the constraints of the maximization problem, and to initiate the simplex algorithm for this problem, artificial variables needed to be introduced. As the reader may have guessed, these artificial variable columns provide the data for the optimal solution point of the dual. Indeed, the required value for the dual objective function of 560 is attained at the point $(\frac{2}{3}, -\frac{1}{3})$, a feasible solution point to the dual, as the reader may confirm.

In general, when solving a *maximization* problem containing equality constraints, the coordinates of the unrestricted dual variables at an optimal solution point to the dual are in the bottom row of the final tableau resolution of the maximization problem in the corresponding artificial variable columns. Problem 12 addresses the proof of this statement.

However, when solving a *minimization* problem containing equality constraints, a sign change adjustment is necessary when determining an optimal solution point to the dual. The value of each unrestricted variable in the optimal solution point to the dual is the *negative* of the entry in the bottom row of the associated artificial variable column. (Why this difference, one might ask? But note that the situations are not identical. For example, our algorithm has been designed for minimization problems; for such a problem, the coefficients of the objective function are entered directly into the initial tableau. To adapt the algorithm to a maximization problem, the corresponding minimization problem is considered, which necessitates an initial sign change in the objective function coefficients when entered into the initial tableau.)

Example 4.4.6. The reduced tableaux resolution for the problem of

$$\begin{aligned}
&\text{Minimizing } z = 16x_1 + 32x_2 + 12x_3\\
&\text{subject to}\\
&\quad x_1 + 5x_2 + x_3 \geq 2\\
&4x_1 + 4x_2 - 2x_3 = 1\\
&x_1, x_2, x_3 \geq 0
\end{aligned}$$

is in Table 4.7. We have Min $z = 14$ attained at $(0, \frac{5}{14}, \frac{3}{14})$. The dual problem is to

Table 4.7

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	1	5	1	−1	1	0	2
x_6	4	4	−2	0	0	1	1
	16	32	12	0	0	0	0
	−5	−9	1	1	0	0	−3
x_3	$-\frac{8}{7}$	0	1	$-\frac{2}{7}$	$\frac{2}{7}$	$-\frac{5}{14}$	$\frac{3}{14}$
x_2	$\frac{3}{7}$	1	0	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{5}{14}$
	16	0	0	8	−8	2	−14

$$\begin{aligned}&\text{Maximize } v = 2y_1 + y_2\\ &\text{subject to}\\ &\ y_1 + 4y_2 \le 16\\ &5y_1 + 4y_2 \le 32\\ &\ y_1 - 2y_2 \le 12\\ &y_1 \ge 0,\ y_2 \text{ unrestricted}\end{aligned}$$

From the final tableau, the point $y_1 = 8$ (using the slack variable x_4 column) and $y_2 = -(2) = -2$ (using the artificial variable x_6 column) is an optimal solution point of the dual, as the reader may easily verify.

Problem Set 4.4

1. Prove Corollary 4.4.3.
2. Show that both the following linear programming problem and its dual do not have any feasible solutions.

$$\begin{aligned}&\text{Maximize } x_1\\ &\text{subject to}\\ &\ x_1 - x_2 \le 1\\ &-x_1 + x_2 \le -2\\ &x_1, x_2 \ge 0\end{aligned}$$

3. Consider the linear programming problem of

$$\begin{aligned}&\text{Maximizing } 4x_1 + 10x_2 - 3x_3 + 2x_4\\ &\text{subject to}\\ &\ 3x_1 - 2x_2 + 7x_3 + x_4 \le 26\\ &\ x_1 + 6x_2 - x_3 + 5x_4 \le 30\\ &-4x_1 + 8x_2 - 2x_3 - x_4 \le 10\\ &x_1, x_2, x_3, x_4 \ge 0\end{aligned}$$

   (a) Show that $(\frac{54}{5}, \frac{16}{5}, 0, 0)$ is a feasible solution to this problem. Compute the value of the objective function at this point.
   (b) Write out the dual problem. Show that $(\frac{7}{10}, \frac{19}{10}, 0)$ is a feasible solution to this problem. What is the value of the objective function of the dual at this point?
   (c) Using Corollary 4.4.2, what can you conclude?

4. Verify that $(0, 5\frac{2}{3}, 8\frac{1}{3}, \frac{1}{3})$ is an optimal solution point to the problem of

$$\begin{array}{l}
\text{Minimizing } 7x_1 + 11x_2 - 3x_3 - x_4 \\
\text{subject to} \\
\begin{array}{rcrcrcrcr}
2x_1 & + & 2x_2 & - & x_3 & - & 3x_4 & \geq & 2 \\
-x_1 & + & 5x_2 & - & 2x_3 & + & x_4 & \geq & 12 \\
x_1 & - & 4x_2 & + & 3x_3 & + & 5x_4 & \geq & 4
\end{array} \\
x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

and $(3\frac{1}{2}, 2, 1\frac{1}{2})$ is an optimal solution point to the dual.

5. Verify that $(0, 3, 2\frac{1}{7}, 0, 1\frac{2}{7})$ is an optimal solution point to the problem of

$$\begin{array}{l}
\text{Maximizing } 3x_1 + 2x_2 + 5x_3 - 2x_4 + x_5 \\
\text{subject to} \\
\begin{array}{rcrcrcrcrcr}
4x_1 & + & x_2 & - & x_3 & + & 2x_4 & + & 4x_5 & \leq & 6 \\
3x_1 & + & 3x_2 & + & 2x_3 & - & x_4 & - & x_5 & \leq & 12 \\
x_1 & - & 2x_2 & + & 5x_3 & - & x_4 & + & x_5 & \leq & 6
\end{array} \\
x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}$$

and that $(\frac{1}{3}, 1, \frac{2}{3})$ is an optimal solution point to the dual.

6. Consider the problem of

$$\begin{array}{l}
\text{Minimizing } z = 13x_1 + 15x_2 + 12x_3 + 8x_4 \\
\text{subject to} \\
\begin{array}{rcrcrcrcr}
4x_1 & + & 8x_2 & - & 5x_3 & + & 3x_4 & = & 32 \\
3x_1 & - & 2x_2 & + & 6x_3 & - & x_4 & \geq & 3
\end{array} \\
x_1, x_2, x_3, x_4 \geq 0
\end{array}$$

   (a) Determine which of the following points are feasible solutions to this min problem: $(9,0,2,2)$, $(4,1,-1,1)$, and $(5,1,1,3)$.
   (b) Evaluate the function z at those points in part (a) that are feasible solutions to the problem.
   (c) Write out the dual to the min problem.
   (d) Determine which of the following points are feasible solutions to this dual problem: $(-1,1)$, $(0,2)$, and $(1,3)$.
   (e) Evaluate the dual objective function at those points in part (d) that are feasible solutions to the problem.
   (f) Using the information above, and only this information, what can you say about the minimum value of z?

7. Solve the following by applying the simplex algorithm to the dual:

$$\begin{array}{l}
\text{Minimize } 8x_1 + 13x_2 + 20x_3 \\
\text{subject to} \\
\begin{array}{rcrcrcr}
3x_1 & + & 2x_2 & + & x_3 & \geq & 2 \\
x_1 & - & x_2 & + & 2x_3 & \geq & 4 \\
 & & 2x_2 & + & 2x_3 & \geq & -1 \\
-2x_1 & + & 3x_2 & & & \geq & 0 \\
4x_1 & & & - & x_3 & \geq & -2
\end{array} \\
x_1, x_2, x_3 \geq 0
\end{array}$$

8. *Generalization of the proof of the Duality Theorem.* Suppose some of the constant terms b_j in (4.4.1) are negative. By rearranging the constraining equations if necessary, assume that $b_i < 0$ for $1 \leq i \leq k$ and $b_i \geq 0$ for $k+1 \leq i \leq m$. Then, to apply the simplex method to (4.4.1), the first k equations must be multiplied by (-1), resulting in all nonnegative terms in the right column. However, now an initial basic feasible solution may not be apparent; if not, artificial variables must be introduced and the simplex method initiated at stage one. Thus the initial tableau would look something like the following:

x_1		x_n	x_{n+1}		x_{n+k}	x_{n+k+1}		x_{n+m}	*Art. Vars.*			
$-a_{11}$	$\dots$	$-a_{1n}$	-1	$\dots$	0	0	$\dots$	0	1	$\dots$	0	$-b_1$
$\vdots$												
$-a_{k1}$	$\dots$	$-a_{kn}$	0	$\dots$	-1	0	$\dots$	0	0	$\dots$	1	$-b_k$
$a_{k+1,1}$	$\dots$	$a_{k+1,n}$	0	$\dots$	0	1	$\dots$	0	0	$\dots$	0	b_{k+1}
$\vdots$												
$a_{m,1}$	$\dots$	$a_{m,n}$	0	$\dots$	0	0	$\dots$	1	0	$\dots$	0	b_m
$-c_1$	$\dots$	$-c_n$	0	$\dots$	0	0	$\dots$	0				0

Since we have assumed that the problem of (4.4.1) has feasible solutions, the simplex method initiated on the above tableau will first drive the artificial variables from the basis and then drive to the optimal value of the objective function. Let r_j, s_i, and t_i be defined as in the proof of the Duality Theorem for $1 \leq j \leq n$ and $1 \leq i \leq m$. Show that the proof given there can be extended to this case, with the only difference being that here $s_i = -t_i$ for $1 \leq i \leq k$.

9. Show that the r_j's as defined in the proof of the Duality Theorem measure the slack in the constraints of the dual problem at the $Y_0 = (s_1, s_2, \dots, s_n)^t$ solution point.

10. The simplex algorithm has been used to resolve the following problems, and the corresponding initial and final tableaux are given (with the w row omitted). For each, construct the dual, determine an optimal solution point to the dual using the data from the tableaux, and verify that your solution point is feasible and optimal.

(a) Minimize $100x_1 + 150x_2$
subject to
$2x_1 + x_2 \geq 13$
$6x_1 - 9x_2 \leq 2$
$7x_1 - 8x_2 \geq 5$
$x_1, x_2 \geq 0$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_6	2	1	-1	0	0	1	0	13
x_4	6	-9	0	1	0	0	0	2
x_7	7	-8	0	0	-1	0	1	5
	100	150	0	0	0	0	0	0
x_5	0	0	$-\frac{5}{8}$	$\frac{23}{24}$	1	$\frac{5}{8}$	-1	$\frac{121}{24}$
x_1	1	0	$-\frac{3}{8}$	$\frac{1}{24}$	0	$\frac{3}{8}$	0	$\frac{119}{24}$
x_2	0	1	$-\frac{1}{4}$	$-\frac{1}{12}$	0	$\frac{1}{4}$	0	$\frac{37}{12}$
	0	0	75	$\frac{25}{3}$	0	-75	0	$-\frac{2875}{3}$

(b) Maximize $3x_1 - 4x_2 + 5x_3$
subject to
$4x_1 - x_2 + 6x_3 \leq 9$
$x_1 + 2x_2 - x_3 = 54$
$x_1, x_2, x_3 \geq 0$

	x_1	x_2	x_3	x_4	x_5	
x_4	4	-1	6	1	0	9
x_5	1	2	-1	0	1	54
	-3	4	-5	0	0	0
x_1	1	0	$\frac{11}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	8
x_2	0	1	$-\frac{10}{9}$	$-\frac{1}{9}$	$\frac{4}{9}$	23
	0	0	$\frac{28}{9}$	$\frac{10}{9}$	$-\frac{13}{9}$	-68

(c) Minimize $-2x_1 + 5x_2 + 9x_3$
subject to
$2x_2 + 5x_3 \geq 1$
$3x_1 - x_2 - x_3 \leq 6$
$2x_1 - 4x_2 + x_3 = 3$
$x_1, x_2, x_3 \geq 0$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_6	0	2	5	-1	0	1	0	1
x_5	3	-1	-1	0	1	0	0	6
x_7	2	-4	1	0	0	0	1	3
	-2	5	9	0	0	0	0	0
x_3	0	0	1	$-\frac{1}{6}$	$-\frac{1}{15}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$
x_2	0	1	0	$-\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$-\frac{1}{4}$	$\frac{1}{3}$
x_1	1	0	0	$-\frac{1}{12}$	$\frac{11}{30}$	$\frac{1}{12}$	$-\frac{1}{20}$	$\frac{32}{15}$
	0	0	0	$\frac{7}{4}$	$\frac{1}{2}$	$-\frac{7}{4}$	$\frac{1}{4}$	2

(d) Minimize $10x_1 + 20x_2 + 15x_3 + 21x_4 + 5x_5$
subject to
$7x_1 - 10x_2 + 8x_3 - 5x_4 + 3x_5 = 730$
$3x_1 + x_2 + 4x_3 - 2x_4 - x_5 = 350$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_6	7	-10	8	-5	3	1	0	730
x_7	3	1	4	2	-1	0	1	350
	10	20	15	21	5	0	0	0
x_1	1	-12	0	-9	5	1	-2	30
x_3	0	$\frac{37}{4}$	1	$\frac{29}{4}$	-4	$-\frac{3}{4}$	$\frac{7}{4}$	65
	0	$\frac{5}{4}$	0	$\frac{9}{4}$	15	$\frac{5}{4}$	$-\frac{25}{4}$	-1275

(e) Maximize $10x_1 - 12x_2 + 11x_3$
subject to
$6x_1 - 7x_2 + 8x_3 = 90$
$-x_1 + 3x_3 \geq 42$
$x_1, x_2, x_3 \geq 0$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	6	-7	8	0	1	0	90
x_6	-1	0	3	-1	0	1	42
	-10	12	-11	0	0	0	0
x_3	$-\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	14
x_2	$-\frac{26}{21}$	1	0	$-\frac{8}{21}$	$-\frac{1}{7}$	$\frac{8}{21}$	$\frac{22}{7}$
	$\frac{25}{21}$	0	0	$\frac{19}{21}$	$\frac{12}{7}$	$-\frac{19}{21}$	$\frac{814}{7}$

11. (a) Consider the linear programming problem of

$$\begin{aligned}&\text{Minimizing } c_1x_1+c_2x_2+\cdots+c_nx_n\\&\text{subject to}\\&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1\\&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2\\&\quad\vdots\\&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m\\&x_1,x_2,\ldots,x_n \geq 0\end{aligned}$$

Assume that $b_i \geq 0$, $1 \leq i \leq m$, and that the problem has a finite optimal solution. To find this solution, suppose the simplex method is used, first adding m slack variables to the problem (each with coefficient (-1)) and then m artificial variables. Let $s_1,s_2,\ldots,s_m$ denote the m entries in the bottom row of the final tableau in the m slack variable columns. Show that $(s_1,s_2,\ldots,s_m)$ is an optimal solution point to the dual, modeling your proof on the proof of the Duality Theorem.

(b) Show that the above result also follows from Problem 8.

12. Consider the linear programming problem of

$$\begin{aligned}&\text{Maximizing } c_1x_1+c_2x_2+\cdots+c_nx_n\\&\text{subject to}\\&a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1\\&\quad\vdots\\&a_{k1}x_1 + \ldots + a_{kn}x_n \leq b_k\\&a_{k+1,1}x_1 + \ldots + a_{k+1n}x_n = b_{k+1}\\&\quad\vdots\\&a_{m1}x_1 + \ldots + a_{mn}x_n = b_m\\&x_1,x_2,\ldots,x_n \geq 0\end{aligned}$$

Assume that $b_i \geq 0$ for $1 \leq i \leq m$. Suppose k slack variables and $m-k$ artificial variables are added to the problem and the simplex algorithm is applied, driving to a finite optimal solution. Denote by s_i, $1 \leq i \leq m$, the entries in the bottom row (the z row) of the final tableau in the slack variable $(1 \leq i \leq k)$ and artificial variable $(k+1 \leq i \leq m)$ columns. Show that $(s_1,s_2,\ldots,s_m)$ is an optimal solution point to the dual.

4.5 THE COMPLEMENTARY SLACKNESS THEOREM

In this section we discuss the Complementary Slackness Theorem. The theorem relates optimal solution points of a linear programming problem and its dual. The theorem will not be needed in any further theoretical developments in the text. However, the relationships prescribed by the theorem are certainly interesting and useful, and will be referred to occasionally in the problem sets and in the development of the transportation problem algorithm in Chapter 7. Those readers who continue their studies in linear programming at a more advanced level may well encounter complementary slackness again.

In Example 4.4.1 of the previous section, it was verified that the point $(10,9,0,0)$ is an optimal solution point to the problem of

$$\begin{array}{l} \text{Maximizing } f(x_1,x_2,x_3,x_4) = -5x_1+18x_2+6x_3-x_4 \\ \text{subject to} \\ \begin{array}{rcrcrcrcl} 2x_1 & & & - & x_3 & + & 3x_4 & \leq & 20 \\ & & x_2 & - & 2x_3 & - & x_4 & \leq & 30 \\ -3x_1 & + & 6x_2 & + & 3x_3 & + & 4x_4 & \leq & 24 \end{array} \\ x_1,x_2,x_3,x_4 \geq 0 \end{array} \tag{4.5.1}$$

and the point $(2,0,3)$ is an optimal solution point to the dual,

$$\begin{array}{l} \text{Minimize } g(y_1,y_2,y_3) = 20y_1+30y_2+24y_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} 2y_1 & & & - & 3y_3 & \geq & -5 \\ & & y_2 & + & 6y_3 & \geq & 18 \\ -y_1 & - & 2y_2 & + & 3y_3 & \geq & 6 \\ 3y_1 & - & y_2 & + & 4y_3 & \geq & -1 \end{array} \\ y_1,y_2,y_3 \geq 0 \end{array} \tag{4.5.2}$$

Since $(10,9,0,0)$ is an optimal solution to (4.5.1), it certainly satisfies the constraints of (4.5.1). In fact, evaluating the three constraints at this point, we find slack of 0, 21, and 0 at the first, second, and third inequalities, respectively. Now the three dual variables y_1,y_2,y_3 of (4.5.2) correspond to these three constraints; and note that where there is positive slack in the constraints of (4.5.1) at the point $(10,9,0,0)$, the value of the corresponding dual variable at the optimal solution point $(2,0,3)$ is 0.

Conversely, evaluating the four constraints of (4.5.2) at $(2,0,3)$ yields slack of 0, 0, 1, and 19. Again, for each inequality at which the slack is positive, the value of the corresponding dual variable at the optimal solution point $(10,9,0,0)$ is 0.

These results are guaranteed by the Complementary Slackness Theorem. Moreover, the converse is also true. In terms of (4.5.1) and (4.5.2), this means that if X^* and Y^* are feasible solutions to (4.5.1) and (4.5.2), respectively, and satisfy the complementary slackness conditions described, they are optimal solution points to their respective problems.

The statement and proof of the general theorem follow.

Theorem 4.5.1 (Complementary Slackness Theorem). *Suppose* $X^* = (x_1^*,\ldots,x_n^*)$ *is a feasible solution to the problem of*

$$\textit{Maximizing } c \cdot X \textit{ subject to } AX \leq b, X \geq 0 \tag{4.5.3}$$

and $Y^* = (y_1^*, \ldots, y_m^*)$ *is a feasible solution to the dual problem of*

$$\textit{Minimizing } b \cdot Y \textit{ subject to } A^t Y \geq c, Y \geq 0 \tag{4.5.4}$$

Then X^* *and* Y^* *are optimal solution points to their respective problems if and only if, for each* i, $1 \leq i \leq m$, *either*

$$\textit{(slack in the ith constraint of (4.5.3) evaluated at } X^*\textit{)} = b_i - \sum_j a_{ij} x_j^* = 0$$

or

$$y_i^* = 0$$

and, for each j, $1 \leq j \leq n$, *either*

$$\textit{(slack in the jth constraint of (4.5.4) evaluated at } Y^*\textit{)} = \sum_i a_{ij} y_i^* - c_j = 0$$

or

$$x_j^* = 0$$

Proof. Corollary 4.4.1 says it all, essentially. Since X^* is a feasible solution to the max problem (4.5.3) and Y^* is a feasible solution to the min problem (4.5.4), from the corollary we have

$$b \cdot Y^* - c \cdot X^* = (b - AX^*) \cdot Y^* + (A^t Y^* - c) \cdot X^*$$

If X^* and Y^* satisfy the complementary slackness hypothesis, then for each i, with $1 \leq i \leq n$, the product

$$y_i^* \left(b_i - \sum\nolimits_j a_{ij} x_j^* \right) = 0$$

that is, each multiplication in the dot product $(b - AX^t) \cdot Y^*$ equals 0, and so $(b - AX^t) \cdot Y^* = 0$. Similarly, from complementary slackness, $(A^t Y^* - c) \cdot X^* = 0$. Thus $b \cdot Y^* = c \cdot X^*$, and so, from Corollary 4.4.2, X^* and Y^* are optimal solution points for their respective problems.

Conversely, if X^* and Y^* are optimal solution points for their respective problems, we have

$$0 = b \cdot Y^* - c \cdot X^* = (b - AX^*) \cdot Y^* + (A^t Y^* - c) \cdot X^*$$

But each dot product on the right side of the equation consists of a sum of products of nonnegative numbers, and so each dot product is nonnegative. Hence both $(b - AX^*) \cdot Y^* = 0$ and $(A^t Y^* - c) \cdot X^* = 0$, that is, the points X^* and Y^* satisfy the complementary slackness conditions. ☐

Example 4.5.1. The problem of

$$\begin{aligned}
&\text{Minimizing } 12x_1 + 5x_2 + 10x_3 \\
&\text{subject to} \\
&\quad x_1 - x_2 + 2x_3 \geq 10 \\
&-3x_1 + x_2 + 4x_3 \geq -9 \\
&\ -x_1 + 2x_2 + 3x_3 \geq 1 \\
&\quad 2x_1 - 3x_2 \geq -2 \\
&\quad 7x_1 - x_2 - 5x_3 \geq 34 \\
&x_1, x_2, x_3 \geq 0
\end{aligned} \tag{4.5.5}$$

has $(7,0,3)$ as an optimal solution point. To determine an optimal solution point to the dual,

$$\begin{aligned}
&\text{Maximize } 10y_1 - 9y_2 + y_3 - 2y_4 + 34y_5 \\
&\text{subject to} \\
&\ y_1 - 3y_2 - y_3 + 2y_4 + 7y_5 \leq 12 \\
&-y_1 + y_2 + 2y_3 - 3y_4 - y_5 \leq 5 \\
&\ 2y_1 + 4y_2 + 3y_3 - 5y_5 \leq 10 \\
&y_1, y_2, y_3, y_4, y_5 \geq 0
\end{aligned} \tag{4.5.6}$$

we can use complementary slackness. Evaluating the inequalities of (4.5.5) at the point $(7,0,3)$, we find positive slack in the first, third, and fourth constraints (and zero slack in the other two). Thus any optimal solution $Y^* = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*)$ to (4.5.6) must have $y_1^* = y_3^* = y_4^* = 0$. And the first and third components of $(7,0,3)$ positive implies that Y^* must yield zero slack in the first and third constraints of (4.5.6). Hence $Y^* = (0, y_2^*, 0, 0, y_5^*)$ and

$$\begin{aligned}
-3y_2^* + 7y_5^* &= 12 \\
4y_2^* - 5y_5^* &= 10
\end{aligned} \tag{4.5.7}$$

The (unique) solution to (4.5.7) is $y_2^* = 10$, $y_5^* = 6$, and so $Y^* = (0, 10, 0, 0, 6)$ is an (and the only) optimal solution point to (4.5.6). (In fact, the existence of this feasible solution to (4.5.6) satisfying complementary slackness now certifies the optimality of $(7,0,3)$.)

Example 4.5.2. Suppose it is claimed that the point $(3,0,1,0)$ is an optimal solution to the problem of

$$\begin{aligned}
&\text{Maximizing } 9x_1 + 3x_2 + 5x_3 + 22x_4 \\
&\text{subject to} \\
&2x_1 - x_2 + 2x_3 + 6x_4 \leq 8 \\
&5x_1 + 3x_2 + x_3 + 2x_4 \leq 16 \\
&4x_1 + x_2 - x_3 + 3x_4 \leq 12 \\
&x_1, x_2, x_3, x_4 \geq 0
\end{aligned} \tag{4.5.8}$$

We can use complementary slackness to attempt to ratify the claim. First, we verify that $(3,0,1,0)$ is a feasible solution to (4.5.8), noting that the point yields zero slack in the first two constraints of (4.5.8) and positive slack in the third. Now consider the dual.

$$\begin{aligned}&\text{Minimize } 8y_1+16y_2+12y_3\\&\text{subject to}\\&2y_1+5y_2+4y_3\geq 9\\&-y_1+3y_2+y_3\geq 3\\&2y_1+y_2-y_3\geq 5\\&6y_1+2y_2+3y_3\geq 22\\&y_1,y_2,y_3\geq 0\end{aligned}\tag{4.5.9}$$

If (4.5.8) has a finite optimal solution, so does (4.5.9), and any optimal solution point $Y^*=(y_1^*,y_2^*,y_3^*)$ must satisfy the complementary slackness conditions with $(3,0,1,0)$. Thus, $y_3^*=0$, and

$$\begin{aligned}2y_1^*+5y_2^*&=9\\2y_1^*+y_2^*&=5\end{aligned}$$

yielding $Y^*=(2,1,0)$. But this point is not a feasible solution to (4.5.9), as the reader may verify. Hence $(3,0,1,0)$ cannot be an optimal solution to (4.5.8).

Problem Set 4.5

1. Consider the linear programming problem of

$$\begin{aligned}&\text{Maximizing } x_1+2x_2\\&\text{subject to}\\&2x_1+x_2\leq 3\\&x_1+2x_2\leq 3\\&x_1,x_2\geq 0\end{aligned}$$

(a) Determine the dual problem.

(b) Show that $X^*=(1,1)$ and $Y^*=(0,1)$ are optimal solutions for the original and dual problems, respectively, by using the Complementary Slackness Theorem.

(c) Note that at these solution points, both y_1^* and the slack in the corresponding first constraint of the max problem are zero.

2. Consider the linear programming problem of

$$\begin{aligned}&\text{Maximizing } 2x_1+2x_2\\&\text{subject to}\\&x_1+x_3+x_4\leq 1\\&x_2+x_3-x_4\leq 1\\&x_1+x_2+2x_3\leq 3\\&x_1,x_2,x_3,x_4\geq 0\end{aligned}$$

(a) Determine the dual problem.
(b) Show that $X^* = (1,1,0,0)$ and $Y^* = (1,1,1)$ are feasible solutions to the original and dual problems, respectively.
(c) Show that for this pair of solutions, for each j, $x_j^* > 0$ implies that the slack in the corresponding dual constraint is zero.
(d) Show that Y^* is not an optimal solution to the dual.
(e) Does this contradict the Complementary Slackness Theorem?

3. Prove or disprove each of the following, using complementary slackness.

(a) $(1,1,0,0)$ is an optimal solution point to the maximization problem of Problem 2.
(b) $(0,4,0,2)$ is an optimal solution point to (4.5.8) on page 157.
(c) $(3,0,1,0,5)$ is an optimal solution point to the problem of

$$\begin{array}{l} \text{Maximizing } 5x_1 + 16x_2 - 4x_3 - x_4 + 7x_5 \\ \text{subject to} \\ 8x_1 - 2x_2 + 3x_3 \qquad\quad - 2x_5 \leq 18 \\ 2x_1 + 4x_2 - 7x_3 + 3x_4 + x_5 \leq 4 \\ x_1 + 3x_2 + x_3 - x_4 + 2x_5 \leq 14 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

(d) $(1,0,1,0)$ is an optimal solution point to the problem of

$$\begin{array}{l} \text{Minimizing } 5x_1 + 8x_2 + 4x_3 + 2x_4 \\ \text{subject to} \\ x_1 + 2x_2 - x_3 + x_4 \geq 0 \\ 2x_1 + 3x_2 + x_3 - x_4 \geq 3 \\ x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

(e) $(0,3,12)$ is an optimal solution point to the problem of

$$\begin{array}{l} \text{Minimizing } 2y_1 - 5y_2 - 3y_3 \\ \text{subject to} \\ -3y_1 - 6y_2 + 2y_3 \geq 6 \\ y_1 + 3y_2 + y_3 \geq 20 \\ 4y_1 + 7y_2 - 3y_3 \geq -15 \\ y_1, y_2, y_3 \geq 0 \end{array}$$

(f) $(0,3,0,0,4)$ is an optimal solution point to the problem of

$$\begin{array}{l} \text{Maximizing } 5x_1 + 4x_2 + 8x_3 + 9x_4 + 15x_5 \\ \text{subject to} \\ x_1 + x_2 + 2x_3 + x_4 + 2x_5 \leq 11 \\ x_1 - 2x_2 - x_3 + 2x_4 + 3x_5 \leq 6 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

Solutions to Selected Problems

Problem Set 2.2

5. There is no change in the optimal solution; all the points of the shaded region in Figure 2.3 satisfy the inequality $4x + 2y \geq 40$.

7. (a) See Example 5.1.1 on page 161.
   (b) There is no change in the optimal diet if $\frac{3}{5} \leq$ the ratio of the cost of Feed 1 to Feed 2 $\leq \frac{5}{3}$.

11. Let x_i denote the amount in pounds of Mineral i used in the production of 100 lb of paint. The problem:

$$\begin{array}{l}
\text{Minimize } 4x_1 + 7.5x_2 + 3x_3 \\
\text{subject to} \\
\begin{array}{rcrcrcr}
 & & 0.06x_2 & + & 0.07x_3 & \geq & 5 \\
0.05x_1 & + & 0.08x_2 & & & \geq & 3 \\
0.30x_1 & + & 0.30x_2 & + & 0.25x_3 & \geq & 26 \\
0.20x_1 & + & 0.10x_2 & + & 0.16x_3 & \leq & 15 \\
x_1 & + & x_2 & + & x_3 & = & 100
\end{array} \\
x_1, x_2, x_3 \geq 0
\end{array}$$

Problem Set 2.3

1. See Example 8.1.1 on page 299.

3. (a) The function to be maximized does not accurately measure profit when less than 2000 lb of aluminum is used.
   (b) The function to be maximized does not accurately measure profit when less than 1500 lb of aluminum is used.
   (c) The first constraint forces the use of at least 1500 lb of aluminum.

5. Replace the function f in (2.3.1) with

$$\begin{aligned}
f(x_1,x_2,x_3,x_4) = {} & 690x_1 + 545x_2 + 1020x_3 + 785x_4 \\
& - 3(35x_1 + 45x_2 + 70x_3 - 2100) \\
& - 2(55x_1 + 42x_2 + 90x_4 - 1800)
\end{aligned}$$

6. Let $x_6 \geq 0$ denote the amount in pounds of Raw Material A purchased and modify the problem of (2.3.2) as follows. Replace the function g with

$$g(x_1,x_2,x_3,x_4,x_5,x_6) = 30x_5 + 690x_1 + 545x_2 + 1020x_3 + 785x_4 + 4x_6$$

An Introduction to Linear Programming and Game Theory, Third Edition. By P. R. Thie and G. E. Keough.

and the second constraint with

$$160x_1 + 100x_2 + 200x_3 + 75x_4 \le 8000 + x_6$$

9. The maximum profit is \$54, attained by making 108 dozen muffins and no brownies.

12. See Example 4.3.2 on page 134.

13. See Problem 4 of Section 4.3 on page 137.

17. Let C, T, B, P, and K denote the number of acres planted of corn, tomatoes, beans, peas, and carrots, respectively; U the number of acres of unused land; L the hours of labor employed; and M the amount of money borrowed. The problem:

$$\begin{array}{l}
\text{Maximize } (60-20)C + 800T + 145B + 185P + 250K - 7.25L - 9U - 0.03M \\
\text{subject to} \\
\quad C + \quad T + \quad B + \quad P + \quad K + \ U \qquad\qquad = 100 \\
\ 5C + 120T + 25B + 35P + 40K + 2U \qquad = L \\
20C + 200T + 55B + 40P + 75K + 9U + 3.25L \le 3000 + M \\
0 \le L \le 3600,\ 0 \le M \le 12000 \\
C, T, B, P, K, U \ge 0
\end{array}$$

Problem Set 2.4

2. (a) See Example 4.3.3 on page 135.

3. Let x_{ij} denote the number of cases shipped from Plant i to Outlet j and x_{i6} the number of surplus cases at Plant i, $1 \le i \le 3$, $1 \le j \le 5$. The problem:

$$\begin{array}{ll}
\text{Minimize} & 6.2x_{11} \qquad\qquad + 5.1x_{13} + 10.1x_{14} + \ 8x_{15} \\
& +6.5x_{21} + 10.5x_{22} + 4.3x_{23} + 11.3x_{24} + 6.5x_{25} \\
& +6.3x_{31} + \ 9x_{32} \qquad\qquad + 10.8x_{34} \\
& -120x_{16} - 110x_{26} - 114x_{36}
\end{array}$$

subject to

$$\begin{array}{l}
\sum_{j=1}^{6} x_{1j} = 4000 \quad (x_{12} = 0) \\
\sum_{j=1}^{6} x_{2j} = 2000 \\
\sum_{j=1}^{6} x_{3j} = 3000 \quad (x_{33} = x_{35} = 0) \\
\sum_{i=1}^{3} x_{ij} = 1000, 1200, 3000, 400, 2200 \quad (j = 1,2,3,4,5, \text{respectively}) \\
x_{i,j} \ge 0
\end{array}$$

Problem Set 2.5

1. Equalities would force each D_i to be at least 1000.

4. For month i ($i = 1$, Aug.; $i = 2$, Sept.; $i = 3$, Oct.), let

$R_i(V_i)$ = number of refrigerators (ovens) bought
$S_i(W_i)$ = number of refrigerators (ovens) sold
$T_i(X_i)$ = number of refrigerators (ovens) stored

The problem:

$$\begin{aligned} \text{Minimize} \quad & 90S_1 + 110S_2 + 105S_3 + 200W_1 + 250W_2 + 240W_3 \\ & -(60R_1 + 65R_2 + 68R_3 + 150V_1 + 175V_2 + 200V_3) \\ & -7(T_1 + T_2 + X_1 + X_2) \end{aligned}$$

subject to

$$\begin{aligned} & 25 + R_1 = S_1 + T_1 & V_1 &= W_1 + X_1 \\ & T_1 + R_2 = S_2 + T_2 & X_1 + V_2 &= W_2 + X_2 \\ & T_2 + R_3 = S_3 & X_2 + V_3 &= W_3 \\ & T_1 + X_1 \leq 45, T_2 + X_2 \leq 45 & & \\ & 0 \leq R_i \leq 65 & 0 \leq V_i &\leq 35 \\ & 0 \leq S_i \leq 100 & 0 \leq W_i &\leq 55 \\ & R_i, S_i, T_i, V_i, W_i, X_i \geq 0 & & \end{aligned}$$

Problem Set 3.1

1. (a) $x_1 = 4, x_2 = 12, x_3 = 0, x_4 = -1$
   (b) Any point $(x_1, x_2, x_3, x_4', x_4'', x_5, x_6)$ of the form $(1, 3, 5, 2 + \lambda, \lambda, 3, 15)$ where $\lambda \geq 0$

3. (a) Minimize $-3x_1 + 2x_2$
   subject to

$$\begin{aligned} 5x_1 + 2x_2 - 3x_3 + x_4 + x_5 \qquad\qquad &= 7 \\ 3x_2 - 4x_3 \qquad + x_6 \qquad &= 6 \\ x_1 \qquad + x_3 - x_4 \qquad\qquad - x_7 &= 11 \\ x_1, \ldots, x_7 \geq 0 & \end{aligned}$$

   (b) Minimize $-x_2' + x_3' - x_3'' + x_4' - x_4''$
   subject to

$$\begin{aligned} x_1 + x_2' \qquad\qquad\qquad\qquad - x_5 \qquad\qquad &= 6 \\ -x_2' + x_3' - x_3'' - x_4' + x_4'' \qquad + x_6 \qquad &= 1 \\ 5x_1 + 6x_2' + 7x_3' - 7x_3'' - 8x_4' + 8x_4'' \qquad\qquad - x_7 &= 2 \\ x_1, x_2', x_3', x_3'', x_4', x_4'', x_5, x_6, x_7 \geq 0 & \end{aligned}$$

(d) Minimize $-6x_1 + 2x_2' - 2x_2'' - 9x_3 - 300$
subject to

$$\begin{array}{rcl} 2x_1 - 6x_2' + 6x_2'' - x_3 + x_4 & = & 100 \\ x_1 + x_2' - x_2'' + 9x_3 + x_5 & = & 200 \\ x_1 + x_6 & = & 50 \\ x_2' - x_2'' - x_7 & = & -60 \\ x_3 - x_8 & = & 5 \end{array}$$

$x_1, x_2', x_2'', x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$

4. (a) $\{(0,0,\lambda,0) : \lambda \geq 11\}$
   (b) $\{(5,0,6,0)\}$
   (c) $\emptyset$

Problem Set 3.2

1. (a) $(1,2,-3)$
   (b) Arbitrarily selecting x_1 and x_2 to use as basic variables, two pivot steps yield the following equivalent system:

$$\begin{array}{rcl} x_2 + \frac{11}{17}x_3 & = & \frac{13}{17} \\ x_1 \quad - \frac{47}{17}x_3 & = & -\frac{3}{17} \end{array}$$

Thus the solution set is

$$\{(-\tfrac{3}{17} + \tfrac{47}{17}\lambda, \tfrac{13}{17} - \tfrac{11}{17}\lambda, \lambda) : \lambda \in \mathbb{R}\}$$

2. The system is equivalent to various systems of equations in canonical form. For example, an equivalent system with basic variables x_1 and x_3 is the system

$$\begin{array}{rcl} x_1 - 8x_2 \quad & = & -41 \\ -3x_2 + x_3 & = & -16 \end{array}$$

4. (a)

$$\begin{array}{rcl} x_2 & = & 9 \\ x_1 - x_3 & = & 4 \end{array}$$

   (b) No
   (c) $b = (17,4)^t$ can be expressed as a linear combination of $A^{(1)} = (2,1)^t$ and $A^{(2)} = (1,0)^t$, but not as a linear combination of $A^{(1)}$ and $A^{(3)} = (-2,-1)^t$

6. (b) $(0,6,2,0)$ and $(0,0,2,2)$

   (d) The minimum value of the objective function is 8, attained at $(0,0,2,2)$

7. Min $f = \frac{15}{4}$ attained at$(\frac{45}{8}, 0, 0, \frac{3}{8})$

Problem Set 3.3

1. (a) $x_1 = 8 - 2x_4, x_2 = 6 - 3x_4, x_3 = 18 - 6x_4$
   (b) $0 \leq x_4 \leq 2$
   (c) x_2
   (d) We should extract x_2 from the basis; therefore, pivot at the $3x_4$ term of the second equation. Pivoting here yields:
   (e)
$$\begin{aligned} x_1 - \tfrac{2}{3}x_2 \qquad &= 4 \\ \tfrac{1}{3}x_2 \qquad + x_4 &= 2 \\ - 2x_2 + x_3 \qquad &= 6 \end{aligned}$$
   The associated basic solution, $(4,0,6,2)$, is feasible.
   (f) The minimum of $\frac{8}{2}$, $\frac{6}{3}$, and $\frac{18}{6}$ is $\frac{6}{3}$, attained with the data from the second equation.

4. Pivoting at the $2x_4$ term of the first constraint yields the equivalent problem of minimizing z with

$$\begin{aligned} \tfrac{1}{2}x_2 - 3x_3 + x_4 &= 3 \\ x_1 + \tfrac{1}{2}x_2 - x_3 \qquad &= 8 \\ 3x_2 - 14x_3 \qquad &= 18 + z \\ x_1, x_2, x_3, x_4 \geq 0 & \end{aligned}$$

The expression for z suggests putting x_3 into the basis, but there is no positive x_3 coefficient in the constraints. In fact, from this representation of the constraints, we see that the set of feasible solutions contains the set

$$\{(8 + x_3, 0, x_3, 3 + 3x_3) : x_3 \geq 0\}$$

What happens to z on this set?

Problem Set 3.4

1. Min $z = -67\frac{1}{3}$ attained at $(0, \frac{97}{3}, 0, \frac{17}{3}, \frac{4}{3})$
2. (a) Min $z = 0$ attained at $(5, 10, 0, 0)$. No pivots necessary.
   (b) Min $z = 0$ attained at $(5, 10, 0, 0)$. No pivots necessary.
   (c) Unbounded objective function.
   (d) Unbounded objective function.
   (e) Min $z = -5$ attained at $(5, 0, 5, 0)$
   (f) Min $z = 0$ attained at $(0, 10, 0, 0)$. One pivot necessary.
   (g) Unbounded objective function. No pivots necessary.
5. When the Min$\{b_i/a_{is} : a_{is} > 0\}$ is attained in more than one row.

Problem Set 3.5

2. (a) Min $z = -200$ attained at $(0, 0, 50, 0)$

(c) Unbounded objective function

(d) Max $z = 90$ attained at $(250, 10, 0, 40, 0, 0)$

3. See Example D.1 on page 427.

5. (a) In the final tableau, $c_2^* = 0$ and at least one $a_{i2}^* > 0$. Thus x_2 can be inserted into the basis. Similarly for x_7.

(b) $(0,0,0,25,0,15,15)$

(c) $(10,30,0,20,0,0,0)$

8. Maximum income is \$7020, attained by producing 240 radios, 85 televisions, and 0 stereos.

Problem Set 3.6

1. (a) Applying the simplex algorithm to the problem of

$$\begin{aligned}&\text{Minimizing } w = x_4 + x_5\\ &\text{subject to}\\ &x_1 - x_2 \qquad\quad + x_4 \qquad = 1\\ &2x_1 + x_2 - x_3 \qquad + x_5 = 3\\ &x_1, x_2, x_3, x_4, x_5 \geq 0\end{aligned}$$

generates the solution point $(\frac{4}{3}, \frac{1}{3}, 0)$ to the original system.

2. (a) Min $z = \frac{3}{2}$ attained at $(0, \frac{29}{2}, \frac{11}{2})$

(b) Min $z = -\frac{36}{5}$ attained at $(0, \frac{21}{5}, \frac{12}{5})$. (Only one artificial variable required.)

(c) No feasible solutions.

4. The row corresponds to the expression for the function $w = x_5 + x_6$ in terms of the nonbasic variables for that tableau, namely, x_2, x_4, x_5, and x_6.

6. Follows from the definition of w and from Problem 9 of Section 3.4 on page 84.

8. Minimal cost is \$1950 attained by using Process 2 for $\frac{3}{2}$ hr and Process 3 for $\frac{9}{2}$ hr.

Problem Set 3.7

3. (a) Min $z = 50$ attained at $(50, 0, 0, 0)$. No redundant equations.

(c) Min $z = -\frac{5}{3}$ attained at $(0, 0, \frac{1}{3}, \frac{5}{3})$. One redundant equation.

(d) Max $z = -6$ attained at $(0, 1, 2, 0)$. No redundant equations.

4. True. If any artificial variables remained in the basis, they would be at zero level. The elimination of these variables from the basis would lead to a degenerate solution to the original system.

Problem Set 3.8

6. (a) Changing the constant-term column entries to 0 in the tableaux of Table 3.4, we have Max $z = 0$ attained at $(0, 0, 0)$.

(b) From the modified tableaux of Table 3.5, the objective function is unbounded.

Problem Set 4.1

1. Maximum gain is \$475, attained at $(25,100)$.
2. (a) Minimum cost is \$475, attained at $(0,\frac{3}{2},\frac{5}{12})$.

Problem Set 4.2

1. (a) Minimize $100y_1+90y_2+500y_3$
   subject to
   $$\begin{aligned} 5y_1 - y_2 \quad &\geq 20 \\ -4y_1 + 12y_2 + y_3 &\geq 30 \\ y_1,y_2,y_3 &\geq 0 \end{aligned}$$

   (b) Maximize $-30y_1-50y_2-80y_3$
   subject to
   $$\begin{aligned} 6y_1 - 2y_2 \quad &\leq 4 \\ 11y_1 + 7y_2 - y_3 &\leq -3 \\ y_1,y_2,y_3 &\geq 0 \end{aligned}$$

   (c) Minimize $60y_1-10y_2+20y_3$
   subject to
   $$\begin{aligned} 5y_1 - 3y_2 + y_3 &\geq -1 \\ y_1 + 8y_2 + 7y_3 &\geq 2 \\ y_1,y_2 \geq 0,&\ y_3 \text{ unrestricted} \end{aligned}$$

   (f) Maximize $50x_1-70x_2-15x_3$
   subject to
   $$\begin{aligned} 4x_1 \quad &\leq 1 \\ 2x_2 \quad &\geq 1 \\ -x_1 - x_2 + x_3 &\geq 4 \\ x_1 \text{ unrestricted},&\ x_2,x_3 \geq 0, \end{aligned}$$

3. (b) Min $b\cdot Y$ is $41\frac{1}{4}$, attained at $(\frac{7}{4},\frac{3}{4})$.
   (c) Max $c\cdot X$ is $41\frac{1}{4}$, attained at $(\frac{25}{8},0,\frac{45}{8})$.

Problem Set 4.5

3. (a) $(1,1,0,0)$ optimal; complementary slackness generates $(2,2,0)$, a feasible solution to the dual.
   (b) $(0,4,0,2)$ optimal; complementary slackness generates $(3,2,0)$, a feasible solution to the dual.

CHAPTER 5

Transportation Model and Its Variants

Chapter Guide. The transportation model is a special class of linear programs that deals with shipping a commodity from *sources* (e.g., factories) to *destinations* (e.g., warehouses). The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits. The application of the transportation model can be extended to other areas of operation, including inventory control, employment scheduling, and personnel assignment.

As you study the material in this chapter, keep in mind that the steps of the transportation algorithm are precisely those of the simplex method. Another point is that the transportation algorithm was developed in the early days of OR to enhance hand computations. Now, with the tremendous power of the computer, such shortcuts may not be warranted and, indeed, are never used in commercial codes in the strict manner presented in this chapter. Nevertheless, the presentation shows that the special transportation tableau is useful in modeling a class of problems in a concise manner (as opposed to the familiar LP model with explicit objective function and constraints). In particular, the transportation tableau format simplifies the solution of the problem by Excel Solver. The representation also provides interesting ideas about how the basic theory of linear programming is exploited to produce shortcuts in computations.

You will find TORA's tutorial module helpful in understanding the details of the transportation algorithm. The module allows you to make the decisions regarding the logic of the computations with immediate feedback.

This chapter includes a summary of 1 real-life application, 12 solved examples, 1 Solver model, 4 AMPL models, 46 end-of-section problems, and 5 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch5Files.

Real-life Application—Scheduling Appointments at Australian Trade Events

The Australian Tourist Commission (ATC) organizes trade events around the world to provide a forum for Australian sellers to meet international buyers of tourism products, including accommodation, tours, and transport. During these events, sellers are

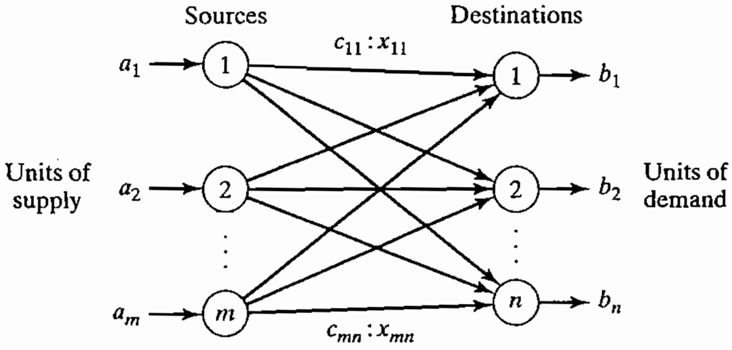


FIGURE 5.1

Representation of the transportation model with nodes and arcs

stationed in booths and are visited by buyers according to scheduled appointments. Because of the limited number of time slots available in each event and the fact that the number of buyers and sellers can be quite large (one such event held in Melbourne in 1997 attracted 620 sellers and 700 buyers), ATC attempts to schedule the seller-buyer appointments in advance of the event in a manner that maximizes preferences. The model has resulted in greater satisfaction for both the buyers and sellers. Case 3 in Chapter 24 on the CD provides the details of the study.

5.1 DEFINITION OF THE TRANSPORTATION MODEL

The general problem is represented by the network in Figure 5.1. There are m sources and n destinations, each represented by a **node**. The **arcs** represent the routes linking the sources and the destinations. Arc (i, j) joining source i to destination j carries two pieces of information: the transportation cost per unit, c_{ij}, and the amount shipped, x_{ij}. The amount of supply at source i is a_i and the amount of demand at destination j is b_j. The objective of the model is to determine the unknowns x_{ij} that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

Example 5.1-1

MG Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars. The mileage chart between the plants and the distribution centers is given in Table 5.1.

The trucking company in charge of transporting the cars charges 8 cents per mile per car. The transportation costs per car on the different routes, rounded to the closest dollar, are given in Table 5.2.

The LP model of the problem is given as

$$\text{Minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

TABLE 5.1 Mileage Chart

	Denver	Miami
Los Angeles	1000	2690
Detroit	1250	1350
New Orleans	1275	850

TABLE 5.2 Transportation Cost per Car

	Denver (1)	Miami (2)
Los Angeles (1)	\$80	\$215
Detroit (2)	\$100	\$108
New Orleans (3)	\$102	\$68

subject to

$$
\begin{array}{lllllll}
x_{11} + x_{12} & & & & & = 1000 & \text{(Los Angeles)} \\
& & x_{21} + x_{22} & & & = 1500 & \text{(Detroit)} \\
& & & + x_{31} & + x_{32} & = 1200 & \text{(New Oreleans)} \\
x_{11} & + x_{21} + & & x_{31} & & = 2300 & \text{(Denver)} \\
& x_{12} + & x_{22} & & + x_{32} & = 1400 & \text{(Miami)} \\
\end{array}
$$

$$x_{ij} \geq 0, i = 1, 2, 3, j = 1, 2$$

These constraints are all equations because the total supply from the three sources (= 1000 + 1500 + 1200 = 3700 cars) equals the total demand at the two destinations (= 2300 + 1400 = 3700 cars).

The LP model can be solved by the simplex method. However, with the special structure of the constraints we can solve the problem more conveniently using the **transportation tableau** shown in Table 5.3.

TABLE 5.3 MG Transportation Model

	Denver	Miami	Supply
Los Angeles	80 x_{11}	215 x_{12}	**1000**
Detroit	100 x_{21}	108 x_{22}	**1500**
New Orleans	102 x_{31}	68 x_{32}	**1200**
Demand	**2300**	**1400**	

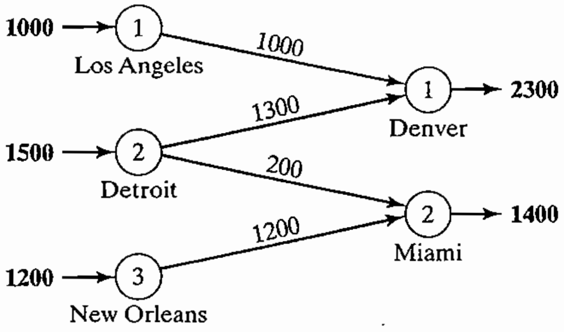


FIGURE 5.2
Optimal solution of MG Auto model

The optimal solution in Figure 5.2 (obtained by TORA[1]) calls for shipping 1000 cars from Los Angeles to Denver, 1300 from Detroit to Denver, 200 from Detroit to Miami, and 1200 from New Orleans to Miami. The associated minimum transportation cost is computed as 1000 × \$80 + 1300 × \$100 + 200 × \$108 + 1200 × \$68 = \$313,200.

Balancing the Transportation Model. The transportation algorithm is based on the assumption that the model is balanced, meaning that the total demand equals the total supply. If the model is unbalanced, we can always add a dummy source or a dummy destination to restore balance.

Example 5.1-2

In the MG model, suppose that the Detroit plant capacity is 1300 cars (instead of 1500). The total supply (= 3500 cars) is less than the total demand (= 3700 cars), meaning that part of the demand at Denver and Miami will not be satisfied.

Because the demand exceeds the supply, a dummy source (plant) with a capacity of 200 cars (= 3700 − 3500) is added to balance the transportation model. The unit transportation costs from the dummy plant to the two destinations are zero because the plant does not exist.

Table 5.4 gives the balanced model together with its optimum solution. The solution shows that the dummy plant ships 200 cars to Miami, which means that Miami will be 200 cars short of satisfying its demand of 1400 cars.

We can make sure that a specific destination does not experience shortage by assigning a very high unit transportation cost from the dummy source to that destination. For example, a penalty of \$1000 in the dummy-Miami cell will prevent shortage at Miami. Of course, we cannot use this "trick" with all the destinations, because shortage must occur somewhere in the system.

The case where the supply exceeds the demand can be demonstrated by assuming that the demand at Denver is 1900 cars only. In this case, we need to add a dummy distribution center to "receive" the surplus supply. Again, the unit transportation costs to the dummy distribution center are zero, unless we require a factory to "ship out" completely. In this case, we must assign a high unit transportation cost from the designated factory to the dummy destination.

---

[1] To use TORA, from Main Menu select Transportation Model. From the SOLVE/MODIFY menu, select Solve ⇒ Final solution to obtain a summary of the optimum solution. A detailed description of the iterative solution of the transportation model is given in Section 5.3.3.

TABLE 5.4 MG Model with Dummy Plant

	Denver	Miami	Supply
Los Angeles	80 **1000**	215	**1000**
Detroit	100 **1300**	108	**1300**
New Orleans	102	68 **1200**	**1200**
Dummy Plant	0	0 **200**	**200**
Demand	**2300**	**1400**	

TABLE 5.5 MG Model with Dummy Destination

	Denver	Miami	Dummy	
Los Angeles	80 **1000**	215	0	**1000**
Detroit	100 **900**	108 **200**	0 **400**	**1500**
New Orleans	102	68 **1200**	0	**1200**
Demand	**1900**	**1400**	**400**	

Table 5.5 gives the new model and its optimal solution (obtained by TORA). The solution shows that the Detroit plant will have a surplus of 400 cars.

PROBLEM SET 5.1A[2]

1. True or False?
   - **(a)** To balance a transportation model, it may be necessary to add both a dummy source and a dummy destination.
   - **(b)** The amounts shipped to a dummy destination represent surplus at the shipping source.
   - **(c)** The amounts shipped from a dummy source represent shortages at the receiving destinations.

---

[2] In this set, you may use TORA to find the optimum solution. AMPL and Solver models for the transportation problem will be introduced at the end of Section 5.3.2.

2. In each of the following cases, determine whether a dummy source or a dummy destination must be added to balance the model.
   (a) Supply: $a_1 = 10, a_2 = 5, a_3 = 4, a_4 = 6$
   Demand: $b_1 = 10, b_2 = 5, b_3 = 7, b_4 = 9$
   (b) Supply: $a_1 = 30, a_2 = 44$
   Demand: $b_1 = 25, b_2 = 30, b_3 = 10$
3. In Table 5.4 of Example 5.1-2, where a dummy plant is added, what does the solution mean when the dummy plant "ships" 150 cars to Denver and 50 cars to Miami?

*4. In Table 5.5 of Example 5.1-2, where a dummy destination is added, suppose that the Detroit plant must ship out *all* its production. How can this restriction be implemented in the model?

5. In Example 5.1-2, suppose that for the case where the demand exceeds the supply (Table 5.4), a penalty is levied at the rate of \$200 and \$300 for each undelivered car at Denver and Miami, respectively. Additionally, no deliveries are made from the Los Angeles plant to the Miami distribution center. Set up the model, and determine the optimal shipping schedule for the problem.

*6. Three electric power plants with capacities of 25, 40, and 30 million kWh supply electricity to three cities. The maximum demands at the three cities are estimated at 30, 35, and 25 million kWh. The price per million kWh at the three cities is given in Table 5.6.

During the month of August, there is a 20% increase in demand at each of the three cities, which can be met by purchasing electricity from another network at a premium rate of \$1000 per million kWh. The network is not linked to city 3, however. The utility company wishes to determine the most economical plan for the distribution and purchase of additional energy.

(a) Formulate the problem as a transportation model.
(b) Determine an optimal distribution plan for the utility company.
(c) Determine the cost of the additional power purchased by each of the three cities.

7. Solve Problem 6, assuming that there is a 10% power transmission loss through the network.
8. Three refineries with daily capacities of 6, 5, and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8, and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is 10 cents per 1000 gallons per pipeline mile. Table 5.7 gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.
   (a) Construct the associated transportation model.
   (b) Determine the optimum shipping schedule in the network.

TABLE 5.6 Price/Million kWh for Problem 6

		City	
	1	2	3
1	\$600	\$700	\$400
Plant 2	\$320	\$300	\$350
3	\$500	\$480	\$450

TABLE 5.7 Mileage Chart for Problem 8

		Distribution area	
	1	2	3
Refinery 1	120	180	—
Refinery 2	300	100	80
Refinery 3	200	250	120

***9.** In Problem 8, suppose that the capacity of refinery 3 is 6 million gallons only and that distribution area 1 must receive all its demand. Additionally, any shortages at areas 2 and 3 will incur a penalty of 5 cents per gallon.

(a) Formulate the problem as a transportation model.

(b) Determine the optimum shipping schedule.

10. In Problem 8, suppose that the daily demand at area 3 drops to 4 million gallons. Surplus production at refineries 1 and 2 is diverted to other distribution areas by truck. The transportation cost per 100 gallons is \$1.50 from refinery 1 and \$2.20 from refinery 2. Refinery 3 can divert its surplus production to other chemical processes within the plant.

(a) Formulate the problem as a transportation model.

(b) Determine the optimum shipping schedule.

11. Three orchards supply crates of oranges to four retailers. The daily demand amounts at the four retailers are 150, 150, 400, and 100 crates, respectively. Supplies at the three orchards are dictated by available regular labor and are estimated at 150, 200, and 250 crates daily. However, both orchards 1 and 2 have indicated that they could supply more crates, if necessary, by using overtime labor. Orchard 3 does not offer this option. The transportation costs per crate from the orchards to the retailers are given in Table 5.8.

(a) Formulate the problem as a transportation model.

(b) Solve the problem.

(c) How many crates should orchards 1 and 2 supply using overtime labor?

12. Cars are shipped from three distribution centers to five dealers. The shipping cost is based on the mileage between the sources and the destinations, and is independent of whether the truck makes the trip with partial or full loads. Table 5.9 summarizes the mileage between the distribution centers and the dealers together with the monthly supply and demand figures given in *number* of cars. A full truckload includes 18 cars. The transportation cost per truck mile is \$25.

(a) Formulate the associated transportation model.

(b) Determine the optimal shipping schedule.

TABLE 5.8 Transportation Cost/Crate for Problem 11

		Retailer		
	1	2	3	4
Orchard 1	\$1	\$2	\$3	\$2
Orchard 2	\$2	\$4	\$1	\$2
Orchard 3	\$1	\$3	\$5	\$3

TABLE 5.9 Mileage Chart and Supply and Demand for Problem 12

			Dealer			
	1	2	3	4	5	Supply
1	100	150	200	140	35	**400**
Center 2	50	70	60	65	80	**200**
3	40	90	100	150	130	**150**
Demand	**100**	**200**	**150**	**160**	**140**	

13. MG Auto, of Example 5.1-1, produces four car models: $M1, M2, M3$, and $M4$. The Detroit plant produces models $M1, M2$, and $M4$. Models $M1$ and $M2$ are also produced in New Orleans. The Los Angeles plant manufactures models $M3$ and $M4$. The capacities of the various plants and the demands at the distribution centers are given in Table 5.10.

The mileage chart is the same as given in Example 5.1-1, and the transportation rate remains at 8 cents per car mile for all models. Additionally, it is possible to satisfy a percentage of the demand for some models from the supply of others according to the specifications in Table 5.11.

(a) Formulate the corresponding transportation model.

(b) Determine the optimum shipping schedule.

(Hint: Add four new destinations corresponding to the new combinations $[M1, M2]$, $[M3, M4]$, $[M1, M2]$, and $[M2, M4]$. The demands at the new destinations are determined from the given percentages.)

TABLE 5.10 Capacities and Demands for Problem 13

	Model				
	$M1$	$M2$	$M3$	$M4$	Totals
Plant					
Los Angeles	—	—	700	300	1000
Detroit	500	600	—	400	1500
New Orleans	800	400	—	—	1200
Distribution center					
Denver	700	500	500	600	2300
Miami	600	500	200	100	1400

TABLE 5.11 Interchangeable Models in Problem 13

Distribution center	Percentage of demand	Interchangeable models
Denver	10	$M1, M2$
	20	$M3, M4$
Miami	10	$M1, M2$
	5	$M2, M4$

5.2 NONTRADITIONAL TRANSPORTATION MODELS

The application of the transportation model is not limited to *transporting* commodities between geographical sources and destinations. This section presents two applications in the areas of production-inventory control and tool sharpening service.

Example 5.2-1 (Production-Inventory Control)

Boralis manufactures backpacks for serious hikers. The demand for its product occurs during March to June of each year. Boralis estimates the demand for the four months to be 100, 200, 180, and 300 units, respectively. The company uses part-time labor to manufacture the backpacks and, accordingly, its production capacity varies monthly. It is estimated that Boralis can produce 50, 180, 280, and 270 units in March through June. Because the production capacity and demand for the different months do not match, a current month's demand may be satisfied in one of three ways.

1. Current month's production.
2. Surplus production in an earlier month.
3. Surplus production in a later month (backordering).

In the first case, the production cost per backpack is $40. The second case incurs an additional holding cost of $.50 per backpack per month. In the third case, an additional penalty cost of $2.00 per backpack is incurred for each month delay. Boralis wishes to determine the optimal production schedule for the four months.

The situation can be modeled as a transportation model by recognizing the following parallels between the elements of the production-inventory problem and the transportation model:

Transportation	Production-inventory
1. Source i	1. Production period i
2. Destination j	2. Demand period j
3. Supply amount at source i	3. Production capacity of period i
4. Demand at destination j	4. Demand for period j
5. Unit transportation cost from source i to destination j	5. Unit cost (production + inventory + penalty) in period i for period j

The resulting transportation model is given in Table 5.12.

TABLE 5.12 Transportation Model for Example 5.2-1

	1	2	3	4	Capacity
1	$40.00	$40.50	$41.00	$41.50	**50**
2	$42.00	$40.00	$40.50	$41.00	**180**
3	$44.00	$42.00	$40.00	$40.50	**280**
4	$46.00	$44.00	$42.00	$40.00	**270**
Demand	**100**	**200**	**180**	**300**	

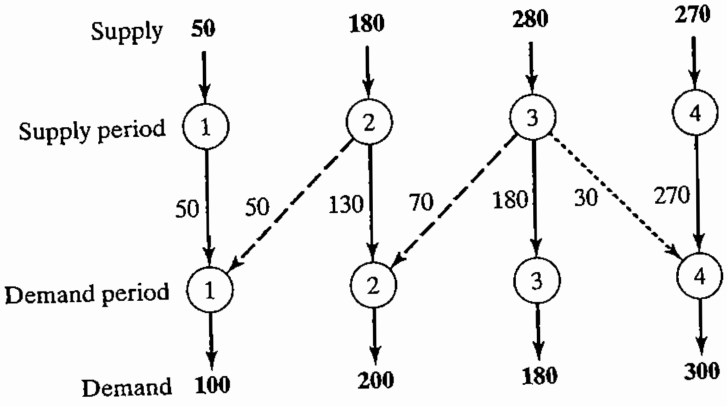


FIGURE 5.3
Optimal solution of the production–inventory model

The unit "transportation" cost from period *i* to period *j* is computed as

$$c_{ij} = \begin{cases} \text{Production cost in } i, i = j \\ \text{Production cost in } i + \text{holding cost from } i \text{ to } j, i < j \\ \text{Production cost in } i + \text{penaty cost from } i \text{ to } j, i > j \end{cases}$$

For example,

$$c_{11} = \$40.00$$

$$c_{24} = \$40.00 + (\$.50 + \$.50) = \$41.00$$

$$c_{41} = \$40.00 + (\$2.00 + \$2.00 + \$2.00) = \$46.00$$

The optimal solution is summarized in Figure 5.3. The dashed lines indicate back-ordering, the dotted lines indicate production for a future period, and the solid lines show production in a period for itself. The total cost is $31,455.

Example 5.2-2 (Tool Sharpening)

Arkansas Pacific operates a medium-sized saw mill. The mill prepares different types of wood that range from soft pine to hard oak according to a weekly schedule. Depending on the type of wood being milled, the demand for sharp blades varies from day to day according to the following 1-week (7-day) data:

Day	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Sun.
Demand (blades)	24	12	14	20	18	14	22

The mill can satisfy the daily demand in the following manner:

1. Buy new blades at the cost of $12 a blade.
2. Use an overnight sharpening service at the cost of $6 a blade.
3. Use a slow 2-day sharpening service at the cost of $3 a blade.

The situation can be represented as a transportation model with eight sources and seven destinations. The destinations represent the 7 days of the week. The sources of the model are defined as follows: Source 1 corresponds to buying new blades, which, in the extreme case, can provide sufficient supply to cover the demand for all 7 days (= 24 + 12 + 14 + 20 + 18 + 14 + 22 = 124). Sources 2 to 8 correspond to the 7 days of the week. The amount of supply for each of these sources equals the number of used blades at the end of the associated day. For example, source 2 (i.e., Monday) will have a supply of used blades equal to the demand for Monday. The unit "transportation cost" for the model is $12, $6, or $3, depending on whether the blade is supplied from new blades, overnight sharpening, or 2-day sharpening. Notice that the overnight service means that used blades sent at the *end* of day i will be available for use at the *start* of day $i + 1$ or day $i + 2$, because the slow 2-day service will not be available until the *start* of day $i + 3$. The "disposal" column is a dummy destination needed to balance the model. The complete model and its solution are given in Table 5.13.

TABLE 5.13 Tool Sharpening Problem Expressed as a Transportation Model

	1 Mon.	2 Tue.	3 Wed.	4 Thu.	5 Fri.	6 Sat.	7 Sun.	8 Disposal	
1-New	$12 24	$12 2	$12	$12	$12	$12	$12	$0 98	124
2-Mon.	M	$6 10	$6 8	$3 6	$3	$3	$3	$0	24
3-Tue.	M	M	$6 6	$6	$3 6	$3	$3	$0	12
4-Wed.	M	M	M	$6 14	$6	$3	$3	$0	14
5-Thu.	M	M	M	M	$6 12	$6	$3 8	$0	20
6-Fri.	M	M	M	M	M	$6 14	$6	$0 4	18
7-Sat.	M	M	M	M	M	M	$6 14	$0	14
8-Sun.	M	M	M	M	M	M	M	$0 22	22
	24	12	14	20	18	14	22	124	

The problem has alternative optima at a cost of $840 (file toraEx5.2-2.txt). The following table summarizes one such solution.

	Number of sharp blades (Target day)			
Period	*New*	*Overnight*	*2-day*	Disposal
Mon.	24 (Mon.)	10 (Tue.) + 8 (Wed.)	6 (Thu.)	0
Tues.	2 (Tue.)	6 (Wed.)	6 (Fri.)	0
Wed.	0	14 (Thu.)	0	0
Thu.	0	12 (Fri.)	8 (Sun.)	0
Fri.	0	14 (Sat.)	0	4
Sat.	0	14 (Sun.)	0	0
Sun.	0	0	0	22

Remarks. The model in Table 5.13 is suitable only for the first week of operation because it does not take into account the *rotational* nature of the days of the week, in the sense that this week's days can act as sources for next week's demand. One way to handle this situation is to assume that the very first week of operation starts with all new blades for each day. From then on, we use a model consisting of exactly 7 sources and 7 destinations corresponding to the days of the week. The new model will be similar to Table 5.13 less source "New" and destination "Disposal." Also, only diagonal cells will be blocked (unit cost $= M$). The remaining cells will have a unit cost of either $3.00 or $6.00. For example, the unit cost for cell (Sat., Mon.) is $6.00 and that for cells (Sat., Tue.), (Sat., Wed.), (Sat., Thu.), and (Sat., Fri.) is $3.00. The table below gives the solution costing $372. As expected, the optimum solution will always use the 2-day service only. The problem has alternative optima (see file toraEx5.2-2a.txt).

	Week $i + 1$							
Week i	*Mon.*	*Tue.*	*Wed.*	*Thu.*	*Fri.*	*Sat.*	*Sun.*	Total
Mon.				6			18	24
Tue.					8		4	12
Wed.	12					2		14
Thu.	8	12						20
Fri.	4		14					18
Sat.				14				14
Sun.					10	12		22
Total	24	12	14	20	18	14	22	

PROBLEM SET 5.2A[3]

1. In Example 5.2-1, suppose that the holding cost per unit is period-dependent and is given by 40, 30, and 70 cents for periods 1, 2, and 3, respectively. The penalty and production costs remain as given in the example. Determine the optimum solution and interpret the results.

---

[3] In this set, you may use TORA to find the optimum solution. AMPL and Solver models for the transportation problem will be introduced at the end of Section 5.3.2.

*2. In Example 5.2-2, suppose that the sharpening service offers 3-day service for $1 a blade on Monday and Tuesday (days 1 and 2). Reformulate the problem, and interpret the optimum solution.

3. In Example 5.2-2, if a blade is not used the day it is sharpened, a holding cost of 50 cents per blade per day is incurred. Reformulate the model, and interpret the optimum solution.

4. JoShop wants to assign four different categories of machines to five types of tasks. The numbers of machines available in the four categories are 25, 30, 20, and 30. The numbers of jobs in the five tasks are 20, 20, 30, 10, and 25. Machine category 4 cannot be assigned to task type 4. Table 5.14 provides the unit cost (in dollars) of assigning a machine category to a task type. The objective of the problem is to determine the optimum number of machines in each category to be assigned to each task type. Solve the problem and interpret the solution.

*5. The demand for a perishable item over the next four months is 400, 300, 420, and 380 tons, respectively. The supply capacities for the same months are 500, 600, 200, and 300 tons. The purchase price per ton varies from month to month and is estimated at $100, $140, $120, and $150, respectively. Because the item is perishable, a current month's supply must be consumed within 3 months (starting with current month). The storage cost per ton per month is $3. The nature of the item does not allow back-ordering. Solve the problem as a transportation model and determine the optimum delivery schedule for the item over the next 4 months.

6. The demand for a special small engine over the next five quarters is 200, 150, 300, 250, and 400 units. The manufacturer supplying the engine has different production capacities estimated at 180, 230, 430, 300, and 300 for the five quarters. Back-ordering is not allowed, but the manufacturer may use overtime to fill the immediate demand, if necessary. The overtime capacity for each period is half the regular capacity. The production costs per unit for the five periods are $100, $96, $116, $102, and $106, respectively. The overtime production cost per engine is 50% higher than the regular production cost. If an engine is produced now for use in later periods, an additional storage cost of $4 per engine per period is incurred. Formulate the problem as a transportation model. Determine the optimum number of engines to be produced during regular time and overtime of each period.

7. Periodic preventive maintenance is carried out on aircraft engines, where an important component must be replaced. The numbers of aircraft scheduled for such maintenance over the next six months are estimated at 200, 180, 300, 198, 230, and 290, respectively. All maintenance work is done during the first day of the month, where a used component may be replaced with a new or an overhauled component. The overhauling of used components may be done in a local repair facility, where they will be ready for use at the beginning of next month, or they may be sent to a central repair shop, where a delay of

TABLE 5.14 Unit Costs for Problem 4

		Task type				
		1	2	3	4	5
Machine category	1	10	2	3	15	9
	2	5	10	15	2	4
	3	15	5	14	7	15
	4	20	15	13	—	8

TABLE 5.15 Bids per Acre for Problem 8

			Location	
		1	2	3
Bidder	1	$520	$210	$570
	2	—	$510	$495
	3	$650	—	$240
	4	$180	$430	$710

3 months (including the month in which maintenance occurs) is expected. The repair cost in the local shop is $120 per component. At the central facility, the cost is only $35 per component. An overhauled component used in a later month will incur an additional storage cost of $1.50 per unit per month. New components may be purchased at $200 each in month 1, with a 5% price increase every 2 months. Formulate the problem as a transportation model, and determine the optimal schedule for satisfying the demand for the component over the next six months.

8. The National Parks Service is receiving four bids for logging at three pine forests in Arkansas. The three locations include 10,000, 20,000, and 30,000 acres. A single bidder can bid for at most 50% of the total acreage available. The bids per acre at the three locations are given in Table 5.15. Bidder 2 does not wish to bid on location 1, and bidder 3 cannot bid on location 2.

(a) In the present situation, we need to *maximize* the total bidding revenue for the Parks Service. Show how the problem can be formulated as a transportation model.

(b) Determine the acreage that should be assigned to each of the four bidders.

5.3 THE TRANSPORTATION ALGORITHM

The transportation algorithm follows the *exact steps* of the simplex method (Chapter 3). However, instead of using the regular simplex tableau, we take advantage of the special structure of the transportation model to organize the computations in a more convenient form.

The special transportation algorithm was developed early on when hand computations were the norm and the shortcuts were warranted. Today, we have powerful computer codes that can solve a transportation model of any size as a regular LP.[4] Nevertheless, the transportation algorithm, aside from its historical significance, does provide insight into the use of the theoretical primal-dual relationships (introduced in Section 4.2) to achieve a practical end result, that of improving hand computations. The exercise is theoretically intriguing.

The details of the algorithm are explained using the following numeric example.

---

[4] In fact, TORA handles all necessary computations in the background using the regular simplex method and uses the transportation model format only as a screen "veneer."

TABLE 5.16 SunRay Transportation Model

	Mill				
	1	2	3	4	Supply
1	10 x_{11}	2 x_{12}	20 x_{13}	11 x_{14}	**15**
Silo 2	12 x_{21}	7 x_{22}	9 x_{23}	20 x_{24}	**25**
3	4 x_{31}	14 x_{32}	16 x_{33}	18 x_{34}	**10**
Demand	**5**	**15**	**15**	**15**	

Example 5.3-1 (SunRay Transport)

SunRay Transport Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (also in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the transportation model in Table 5.16. The unit transportation costs, c_{ij}, (shown in the northeast corner of each box) are in hundreds of dollars. The model seeks the minimum-cost shipping schedule x_{ij} between silo i and mill j $(i = 1, 2, 3; j = 1, 2, 3, 4)$.

Summary of the Transportation Algorithm. The steps of the transportation algorithm are exact parallels of the simplex algorithm.

Step 1. Determine a *starting* basic feasible solution, and go to step 2.

Step 2. Use the optimality condition of the simplex method to determine the *entering variable* from among all the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise, go to step 3.

Step 3. Use the feasibility condition of the simplex method to determine the *leaving variable* from among all the current basic variables, and find the new basic solution. Return to step 2.

5.3.1 Determination of the Starting Solution

A general transportation model with m sources and n destinations has $m + n$ constraint equations, one for each source and each destination. However, because the transportation model is always balanced (sum of the supply = sum of the demand), one of these equations is redundant. Thus, the model has $m + n - 1$ independent constraint equations, which means that the starting basic solution consists of $m + n - 1$ basic variables. Thus, in Example 5.3-1, the starting solution has $3 + 4 - 1 = 6$ basic variables.

The special structure of the transportation problem allows securing a nonartificial starting basic solution using one of three methods:[5]

1. Northwest-corner method
2. Least-cost method
3. Vogel approximation method

The three methods differ in the "quality" of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value. In general, though not always, the Vogel method yields the best starting basic solution, and the northwest-corner method yields the worst. The tradeoff is that the northwest-corner method involves the least amount of computations.

Northwest-Corner Method. The method starts at the northwest-corner cell (route) of the tableau (variable x_{11}).

Step 1. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step 2. Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both a row and a column net to zero simultaneously, *cross out one only*, and leave a zero supply (demand) in the uncrossed-out row (column).

Step 3. If *exactly one* row or column is left uncrossed out, stop. Otherwise, move to the cell to the right if a column has just been crossed out or below if a row has been crossed out. Go to step 1.

Example 5.3-2

The application of the procedure to the model of Example 5.3-1 gives the starting basic solution in Table 5.17. The arrows show the order in which the allocated amounts are generated.

The starting basic solution is

$$x_{11} = 5, x_{12} = 10$$

$$x_{22} = 5, x_{23} = 15, x_{24} = 5$$

$$x_{34} = 10$$

The associated cost of the schedule is

$$z = 5 \times 10 + 10 \times 2 + 5 \times 7 + 15 \times 9 + 5 \times 20 + 10 \times 18 = \$520$$

Least-Cost Method. The least-cost method finds a better starting solution by concentrating on the cheapest routes. The method assigns as much as possible to the cell with the smallest unit cost (ties are broken arbitrarily). Next, the satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.

---

[5] All three methods are featured in TORA's tutorial module. See the end of Section 5.3.3.

TABLE 5.17 Northwest-Corner Starting Solution

	1	2	3	4	Supply
1	10 5	2 10	20	11	**15**
2	12	7 5	9 15	20 5	**25**
3	4	14	16	18 10	**10**
Demand	**5**	**15**	**15**	**15**	

If both a row and a column are satisfied simultaneously, *only one is crossed out*, the same as in the northwest-corner method. Next, look for the uncrossed-out cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed out.

Example 5.3-3

The least-cost method is applied to Example 5.3-1 in the following manner:

1. Cell (1, 2) has the least unit cost in the tableau (= \$2). The most that can be shipped through (1, 2) is $x_{12} = 15$ truckloads, which happens to satisfy both row 1 and column 2 simultaneously. We arbitrarily cross out column 2 and adjust the supply in row 1 to 0.
2. Cell (3, 1) has the smallest uncrossed-out unit cost (= \$4). Assign $x_{31} = 5$, and cross out column 1 because it is satisfied, and adjust the demand of row 3 to 10 − 5 = 5 truckloads.
3. Continuing in the same manner, we successively assign 15 truckloads to cell (2, 3), 0 truckloads to cell (1, 4), 5 truckloads to cell (3, 4), and 10 truckloads to cell (2, 4) (verify!).

The resulting starting solution is summarized in Table 5.18. The arrows show the order in which the allocations are made. The starting solution (consisting of 6 basic variables) is $x_{12} = 15, x_{14} = 0, x_{23} = 15, x_{24} = 10, x_{31} = 5, x_{34} = 5$. The associated objective value is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

The quality of the least-cost starting solution is better than that of the northwest-corner method (Example 5.3-2) because it yields a smaller value of z (\$475 versus \$520 in the northwest-corner method).

Vogel Approximation Method (VAM). VAM is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

Step 1. For each row (column), determine a penalty measure by subtracting the *smallest* unit cost element in the row (column) from the *next smallest* unit cost element in the same row (column).

TABLE 5.18 Least-Cost Starting Solution

	1	2	3	4	Supply
1	10	(start) 2 15	20	11 0	**15**
2	12	7	9 15	(end) 20 10	**25**
3	4 5	14	16	18 5	**10**
Demand	**5**	**15**	**15**	**15**	

Step 2. Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row *or* column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 3. **(a)** If exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) If one row (column) with *positive* supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least-cost method. Stop.

(c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the *zero* basic variables by the least-cost method. Stop.

(d) Otherwise, go to step 1.

Example 5.3-4

VAM is applied to Example 5.3-1. Table 5.19 computes the first set of penalties.

Because row 3 has the largest penalty (= 10) and cell (3, 1) has the smallest unit cost in that row, the amount 5 is assigned to x_{31}. Column 1 is now satisfied and must be crossed out. Next, new penalties are recomputed as in Table 5.20.

Table 5.20 shows that row 1 has the highest penalty (= 9). Hence, we assign the maximum amount possible to cell (1, 2), which yields $x_{12} = 15$ and simultaneously satisfies both row 1 and column 2. We arbitrarily cross out column 2 and adjust the supply in row 1 to zero.

Continuing in the same manner, row 2 will produce the highest penalty (= 11), and we assign $x_{23} = 15$, which crosses out column 3 and leaves 10 units in row 2. Only column 4 is left, and it has a positive supply of 15 units. Applying the least-cost method to that column, we successively assign $x_{14} = 0$, $x_{34} = 5$, and $x_{24} = 10$ (verify!). The associated objective value for this solution is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

This solution happens to have the same objective value as in the least-cost method.

TABLE 5.19 Row and Column Penalties in VAM

	1	2	3	4		Row penalty
1	10	2	20	11	15	10 − 2 = 8
2	12	7	9	20	25	9 − 7 = 2
3	4 5	14	16	18	10	14 − 4 = 10
	5	15	15	15		
Column penalty	10 − 4 = 6	7 − 2 = 5	16 − 9 = 7	18 − 11 = 7		

TABLE 5.20 First Assignment in VAM ($x_{31} = 5$)

	1	2	3	4		Row penalty
1	10	2	20	11	15	9
2	12	7	9	20	25	2
3	4 5	14	16	18	10	2
	5	15	15	15		
Column penalty	—	5	7	7		

PROBLEM SET 5.3A

1. Compare the starting solutions obtained by the northwest-corner, least-cost, and Vogel methods for each of the following models:

*(a)

0	2	1	6
2	1	5	7
2	4	3	7
5	5	10	

(b)

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

(c)

5	1	8	12
2	4	0	14
3	6	7	4
9	10	11	

5.3.2 Iterative Computations of the Transportation Algorithm

After determining the starting solution (using any of the three methods in Section 5.3.1), we use the following algorithm to determine the optimum solution:

Step 1. Use the simplex *optimality condition* to determine the *entering variable* as the current nonbasic variable that can improve the solution. If the optimality condition is satisfied, stop. Otherwise, go to step 2.

Step 2. Determine the *leaving variable* using the simplex *feasibility condition.* Change the basis, and return to step 1.

The optimality and feasibility conditions do not involve the familiar row operations used in the simplex method. Instead, the special structure of the transportation model allows simpler computations.

Example 5.3-5

Solve the transportation model of Example 5.3-1, starting with the northwest-corner solution.

Table 5.21 gives the northwest-corner starting solution as determined in Table 5.17, Example 5.3-2.

The determination of the entering variable from among the current nonbasic variables (those that are not part of the starting basic solution) is done by computing the nonbasic coefficients in the z-row, using the **method of multipliers** (which, as we show in Section 5.3.4, is rooted in LP duality theory).

In the method of multipliers, we associate the multipliers u_i and v_j with row i and column j of the transportation tableau. For each current *basic* variable x_{ij}, these multipliers are shown in Section 5.3.4 to satisfy the following equations:

$$u_i + v_j = c_{ij}, \text{ for each } \textit{basic } x_{ij}$$

As Table 5.21 shows, the starting solution has 6 basic variables, which leads to 6 equations in 7 unknowns. To solve these equations, the method of multipliers calls for arbitrarily setting any $u_i = 0$, and then solving for the remaining variables as shown below.

Basic variable	(u, v) Equation	Solution
x_{11}	$u_1 + v_1 = 10$	Set $u_1 = 0 \rightarrow v_1 = 10$
x_{12}	$u_1 + v_2 = 2$	$u_1 = 0 \rightarrow v_2 = 2$
x_{22}	$u_2 + v_2 = 7$	$v_2 = 2 \rightarrow u_2 = 5$
x_{23}	$u_2 + v_3 = 9$	$u_2 = 5 \rightarrow v_3 = 4$
x_{24}	$u_2 + v_4 = 20$	$u_2 = 5 \rightarrow v_4 = 15$
x_{34}	$u_3 + v_4 = 18$	$v_4 = 15 \rightarrow u_3 = 3$

To summarize, we have

$$u_1 = 0, u_2 = 5, u_3 = 3$$
$$v_1 = 10, v_2 = 2, v_3 = 4, v_4 = 15$$

Next, we use u_i and v_j to evaluate the nonbasic variables by computing

$$u_i + v_j - c_{ij}, \text{ for each } \textit{nonbasic } x_{ij}$$

TABLE 5.21 Starting Iteration

	1	2	3	4	Supply
1	10 **5**	2 **10**	20	11	**15**
2	12	7 **5**	9 **15**	20 **5**	**25**
3	4	14	16	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

The results of these evaluations are shown in the following table:

Nonbasic variable	$u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 4 - 20 = -16$
x_{14}	$u_1 + v_4 - c_{14} = 0 + 15 - 11 = 4$
x_{21}	$u_2 + v_1 - c_{21} = 5 + 10 - 12 = 3$
x_{31}	$u_3 + v_1 - c_{31} = 3 + 10 - 4 = \mathbf{9}$
x_{32}	$u_3 + v_2 - c_{32} = 3 + 2 - 14 = -9$
x_{33}	$u_3 + v_3 - c_{33} = 3 + 4 - 16 = -9$

The preceding information, together with the fact that $u_i + v_j - c_{ij} = 0$ for each basic x_{ij}, is actually equivalent to computing the z-row of the simplex tableau, as the following summary shows.

Basic	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	↓ x_{31}	x_{32}	x_{33}	x_{34}
z	0	0	−16	4	3	0	0	0	9	−9	−9	0

Because the transportation model seeks to *minimize* cost, the entering variable is the one having the *most positive* coefficient in the z-row. Thus, x_{31} is the entering variable.

The preceding computations are usually done directly on the transportation tableau as shown in Table 5.22, meaning that it is not necessary really to write the (u, v)-equations explicitly. Instead, we start by setting $u_1 = 0$.[6] Then we can compute the v-values of all the columns that have *basic* variables in row 1—namely, v_1 and v_2. Next, we compute u_2 based on the (u, v)-equation of basic x_{22}. Now, given u_2, we can compute v_3 and v_4. Finally, we determine u_3 using the basic equation of x_{33}. Once all the u's and v's have been determined, we can evaluate the nonbasic variables by computing $u_i + v_j - c_{ij}$ for each nonbasic x_{ij}. These evaluations are shown in Table 5.22 in the boxed southeast corner of each cell.

Having identified x_{31} as the entering variable, we need to determine the leaving variable. Remember that if x_{31} enters the solution to become basic, one of the current basic variables must leave as nonbasic (at zero level).

TABLE 5.22 Iteration 1 Calculations

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 **5**	2 **10**	20 $\boxed{-16}$	11 $\boxed{4}$	**15**
$u_2 = 5$	12 $\boxed{3}$	7 **5**	9 **15**	20 **5**	**25**
$u_3 = 3$	4 $\boxed{9}$	14 $\boxed{-9}$	16 $\boxed{-9}$	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

---

[6] The tutorial module of TORA is designed to demonstrate that assigning a zero initial value to any u or v does not affect the optimization results. See TORA Moment on page 216.

The selection of x_{31} as the entering variable means that we want to ship through this route because it reduces the total shipping cost. What is the most that we can ship through the new route? Observe in Table 5.22 that if route (3, 1) ships θ units (i.e., $x_{31} = \theta$), then the maximum value of θ is determined based on two conditions.

1. Supply limits and demand requirements remain satisfied.
2. Shipments through all routes remain nonnegative.

These two conditions determine the maximum value of θ and the leaving variable in the following manner. First, construct a *closed loop* that starts and ends at the entering variable cell, (3, 1). The loop consists of *connected horizontal* and *vertical* segments only (no diagonals are allowed).[7] Except for the entering variable cell, each corner of the closed loop must coincide with a basic variable. Table 5.23 shows the loop for x_{31}. Exactly one loop exists for a given entering variable.

Next, we assign the amount θ to the entering variable cell (3, 1). For the supply and demand limits to remain satisfied, we must alternate between subtracting and adding the amount θ at the successive *corners* of the loop as shown in Table 5.23 (it is immaterial whether the loop is traced in a clockwise or counterclockwise direction). For $\theta \geq 0$, the new values of the variables then remain nonnegative if

$$x_{11} = 5 - \theta \geq 0$$

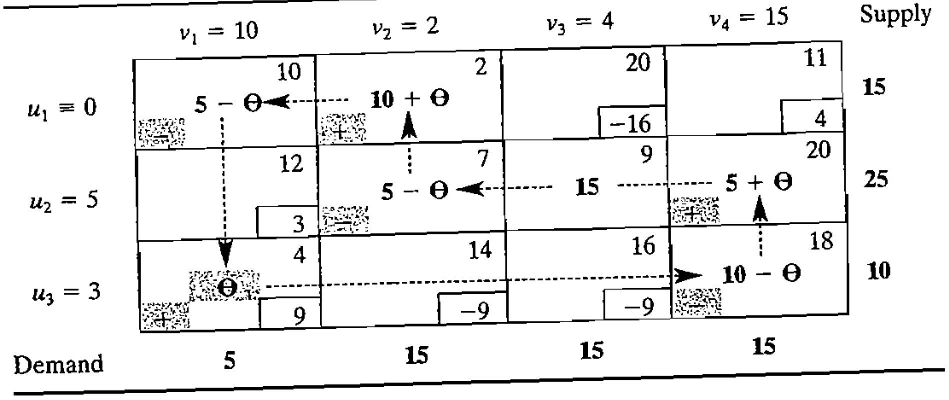
$$x_{22} = 5 - \theta \geq 0$$

$$x_{34} = 10 - \theta \geq 0$$

The corresponding maximum value of θ is 5, which occurs when both x_{11} and x_{22} reach zero level. Because only one current basic variable must leave the basic solution, we can choose either x_{11} or x_{22} as the leaving variable. We arbitrarily choose x_{11} to leave the solution.

The selection of x_{31} (= 5) as the entering variable and x_{11} as the leaving variable requires adjusting the values of the basic variables at the corners of the closed loop as Table 5.24 shows. Because each unit shipped through route (3, 1) reduces the shipping cost by \$9 ($= u_3 + v_1 - c_{31}$), the total cost associated with the new schedule is \$9 × 5 = \$45 less than in the previous schedule. Thus, the new cost is \$520 − \$45 = \$475.

TABLE 5.23 Determination of Closed Loop for x_{31}



	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 $5 - \Theta$ −	2 $10 + \Theta$ +	20 −16	11 4	15
$u_2 = 5$	12 3	7 $5 - \Theta$ −	9 15	20 $5 + \Theta$ +	25
$u_3 = 3$	4 Θ +	14 −9	16 −9	18 $10 - \Theta$ −	10
Demand	5	15	15	15	

---

[7] TORA's tutorial module allows you to determine the cells of the *closed loop* interactively with immediate feedback regarding the validity of your selections. See TORA Moment on page 216.

TABLE 5.24 Iteration 2 Calculations

	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 −9	2 **$15 - \theta$** −	20 −16	11 **θ** + 4	**15**
$u_2 = 5$	12 −6	7 **$0 + \theta$** +	9 **15**	20 **$10 - \theta$** −	**25**
$u_3 = 3$	4 **5**	14 −9	16 −9	18 **5**	**10**
Demand	**5**	**15**	**15**	**15**	

TABLE 5.25 Iteration 3 Calculations (Optimal)

	$v_1 = -3$	$v_2 = 2$	$v_3 = 4$	$v_4 = 11$	Supply
$u_1 \equiv 0$	10 −13	2 **5**	20 −16	11 **10**	**15**
$u_2 = 5$	12 −10	7 **10**	9 **15**	20 −4	**25**
$u_3 = 7$	4 **5**	14 −5	16 −5	18 **5**	**10**
Demand	**5**	**15**	**15**	**15**	

Given the new basic solution, we repeat the computation of the multipliers u and v, as Table 5.24 shows. The entering variable is x_{14}. The closed loop shows that $x_{14} = 10$ and that the leaving variable is x_{24}.

The new solution, shown in Table 5.25, costs \$4 × 10 = \$40 less than the preceding one, thus yielding the new cost \$475 − \$40 = \$435. The new $u_i + v_j - c_{ij}$ are now negative for all nonbasic x_{ij}. Thus, the solution in Table 5.25 is optimal.

The following table summarizes the optimum solution.

From silo	To mill	Number of truckloads
1	2	5
1	4	10
2	2	10
2	3	15
3	1	5
3	4	5
	Optimal cost = \$435	

TORA Moment.

From Solve/Modify Menu, select Solve ⇒ Iterations, and choose one of the three methods (northwest corner, least-cost, or Vogel) to start the transportation model iterations. The iterations module offers two useful interactive features:

1. You can set any u or v to zero before generating Iteration 2 (the default is $u_1 = 0$). Observe then that although the values of u_i and v_j change, the evaluation of the nonbasic cells ($= u_i + v_j - c_{ij}$) remains the same. This means that, initially, any u or v can be set to zero (in fact, any value) without affecting the optimality calculations.
2. You can test your understanding of the selection of the *closed loop* by clicking (in any order) the *corner* cells that comprise the path. If your selection is correct, the cell will change color (green for entering variable, red for leaving variable, and gray otherwise).

Solver Moment.

Entering the transportation model into Excel spreadsheet is straightforward. Figure 5.4 provides the Excel Solver template for Example 5.3-1 (file solverEx5.3-1.xls), together with all the formulas and the definition of range names.

In the input section, data include unit cost matrix (cells B4:E6), source names (cells A4:A6), destination names (cells B3:E3), supply (cells F4:F6), and demand (cells B7:E7). In the output section, cells B11:E13 provide the optimal solution in matrix form. The total cost formula is given in target cell A10.

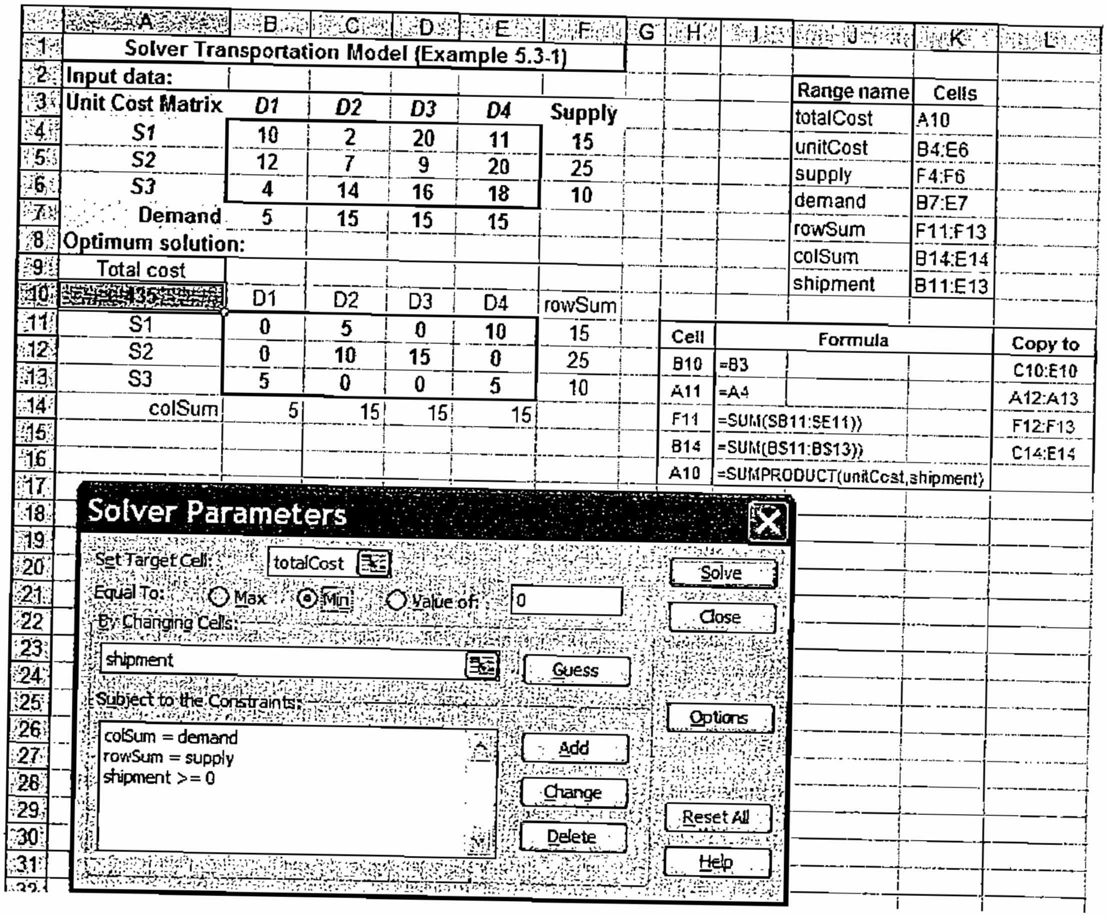
AMPL Moment.

Figure 5.5 provides the AMPL model for the transportation model of Example 5.3-1 (file amplEx5.3-1a.txt). The names used in the model are self-explanatory. Both the constraints and the objective function follow the format of the LP model presented in Example 5.1-1.

The model uses the sets `sNodes` and `dNodes` to conveniently allow the use of the alphanumeric set members `{S1, S2, S3}` and `{D1, D2, D3, D4}` which are entered in the data section. All the input data are then entered in terms of these set members as shown in Figure 5.5.

Although the alphanumeric code for set members is more readable, generating them for large problems may not be convenient. File amplEx5.3-1b shows how the same sets can be defined as `{1..m}` and `{1..n}`, where `m` and `n` represent the number of sources and the number of destinations. By simply assigning numeric values for `m` and `n`, the sets are automatically defined for any size model.

The data of the transportation model can be retrieved from a spreadsheet (file TM.xls) using the AMPL `table` statement. File amplEx3.5-1c.txt provides the details. To study this model, you will need to review the material in Section A.5.5.



	A	B	C	D	E	F
1	**Solver Transportation Model (Example 5.3-1)**					
2	**Input data:**					
3	**Unit Cost Matrix**	***D1***	***D2***	***D3***	***D4***	**Supply**
4	***S1***	10	2	20	11	15
5	***S2***	12	7	9	20	25
6	***S3***	4	14	16	18	10
7	**Demand**	5	15	15	15	
8	**Optimum solution:**					
9	Total cost					
10	435	D1	D2	D3	D4	rowSum
11	S1	0	5	0	10	15
12	S2	0	10	15	0	25
13	S3	5	0	0	5	10
14	colSum	5	15	15	15	

Range name	Cells
totalCost	A10
unitCost	B4:E6
supply	F4:F6
demand	B7:E7
rowSum	F11:F13
colSum	B14:E14
shipment	B11:E13

Cell	Formula	Copy to
B10	=B3	C10:E10
A11	=A4	A12:A13
F11	=SUM($B11:$E11))	F12:F13
B14	=SUM(B$11:B$13))	C14:E14
A10	=SUMPRODUCT(unitCost,shipment)	

Solver Parameters

Set Target Cell: totalCost

Equal To: ○ Max ● Min ○ Value of: 0

By Changing Cells:

shipment

Subject to the Constraints:

colSum = demand
rowSum = supply
shipment >= 0

FIGURE 5.4

Excel Solver solution of the transportation model of Example 5.3-1 (File solverEx5.3-1.xls)

PROBLEM SET 5.3B

1. Consider the transportation models in Table 5.26.
   - **(a)** Use the northwest-corner method to find the starting solution.
   - **(b)** Develop the iterations that lead to the optimum solution.
   - **(c)** *TORA Experiment.* Use TORA's Iterations module to compare the effect of using the northwest-corner rule, least-cost method, and Vogel method on the number of iterations leading to the optimum solution.
   - **(d)** *Solver Experiment.* Solve the problem by modifying file solverEx5.3-1.xls.
   - **(e)** *AMPL Experiment.* Solve the problem by modifying file amplEx5.3-1b.txt.
2. In the transportation problem in Table 5.27, the total demand exceeds the total supply. Suppose that the penalty costs per unit of unsatisfied demand are $5, $3, and $2 for destinations 1, 2, and 3, respectively. Use the least-cost starting solution and compute the iterations leading to the optimum solution.

```
#----- Transporation model (Example 5.3-1)-----
set sNodes;
set dNodes;
param c{sNodes,dNodes};
param supply{sNodes};
param demand{dNodes};
var x{sNodes,dNodes}>=0;
minimize z:sum {i in sNodes,j in dNodes}c[i,j]*x[i,j];
subject to
source {i in sNodes}:sum{j in dNodes}x[i,j]=supply[i];
dest {j in dNodes}:sum{i in sNodes}x[i,j]=demand[j];
data;
set sNodes:=S1 S2 S3;
set dNodes:=D1 D2 D3 D4;
param c:
      D1  D2  D3  D4 :=
S1 10  2   20  11
S2 12  7   9   20
S3 4   14  16  18;
param supply:= S1 15 S2 25 S3 10;
param demand:=D1 5 D2 15 D3 15 D4 15;
solve;display z, x;
```

FIGURE 5.5

AMPL model of the transportation model of Example 5.3-1 (File amplEx5.3-1a.txt)

TABLE 5.26 Transportation Models for Problem 1

(i)				(ii)				(iii)			
\$0	\$2	\$1	**6**	\$10	\$4	\$2	**8**	—	\$3	\$5	**4**
\$2	\$1	\$5	**9**	\$2	\$3	\$4	**5**	\$7	\$4	\$9	**7**
\$2	\$4	\$3	**5**	\$1	\$2	\$0	**6**	\$1	\$8	\$6	**19**
5	**5**	**10**		**7**	**6**	**6**		**5**	**6**	**19**	

TABLE 5.27 Data for Problem 2

\$5	\$1	\$7	**10**
\$6	\$4	\$6	**80**
\$3	\$2	\$5	**15**
75	**20**	**50**	

3. In Problem 2, suppose that there are no penalty costs, but that the demand at destination 3 must be satisfied completely.

- **(a)** Find the optimal solution.
- **(b)** *Solver Experiment.* Solve the problem by modifying file solverEx5.3-1.xls.
- **(c)** *AMPL Experiment.* Solve the problem by modifying file amplEx5.3b-1.txt.

TABLE 5.28 Data for Problem 4

\$1	\$2	\$1	**20**
\$3	\$4	\$5	**40**
\$2	\$3	\$3	**30**
30	**20**	**20**	

TABLE 5.29 Data for Problem 6

10			**10**
	20	**20**	**40**
10	**20**	**20**	

4. In the unbalanced transportation problem in Table 5.28, if a unit from a source is not shipped out (to any of the destinations), a storage cost is incurred at the rate of \$5, \$4, and \$3 per unit for sources 1, 2, and 3, respectively. Additionally, all the supply at source 2 must be shipped out completely to make room for a new product. Use Vogel's starting solution and determine all the iterations leading to the optimum shipping schedule.

*5. In a 3×3 transportation problem, let x_{ij} be the amount shipped from source i to destination j and let c_{ij} be the corresponding transportation cost per unit. The amounts of supply at sources 1, 2, and 3 are 15, 30, and 85 units, respectively, and the demands at destinations 1, 2, and 3 are 20, 30, and 80 units, respectively. Assume that the starting northwest-corner solution is optimal and that the associated values of the multipliers are given as $u_1 = -2, u_2 = 3, u_3 = 5, v_1 = 2, v_2 = 5$, and $v_3 = 10$.
   - (a) Find the associated optimal cost.
   - (b) Determine the smallest value of c_{ij} for each nonbasic variable that will maintain the optimality of the northwest-corner solution.

6. The transportation problem in Table 5.29 gives the indicated *degenerate* basic solution (i.e., at least one of the basic variables is zero). Suppose that the multipliers associated with this solution are $u_1 = 1, u_2 = -1, v_1 = 2, v_2 = 2$, and $v_3 = 5$ and that the unit cost for all (basic and nonbasic) *zero* x_{ij} variables is given by

$$c_{ij} = i + j\theta, -\infty < \theta < \infty$$

   - (a) If the given solution is optimal, determine the associated optimal value of the objective function.
   - (b) Determine the value of θ that will guarantee the optimality of the given solution. (*Hint:* Locate the zero basic variable.)

7. Consider the problem

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

TABLE 5.30 Data for Problem 7

\$1	\$1	\$2	5
\$6	\$5	\$1	6
2	7	1	

subject to

$$\sum_{j=1}^{n} x_{ij} \geq a_i, i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j, j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0, \text{all } i \text{ and } j$$

It may appear logical to assume that the optimum solution will require the first (second) set of inequalities to be replaced with equations if $\sum a_i \geq \sum b_j$ ($\sum a_i \leq \sum b_j$). The counterexample in Table 5.30 shows that this assumption is not correct.

Show that the application of the suggested procedure yields the solution $x_{11} = 2$, $x_{12} = 3$, $x_{22} = 4$, and $x_{23} = 2$, with $z = \$27$, which is worse than the feasible solution $x_{11} = 2$, $x_{12} = 7$, and $x_{23} = 6$, with $z = \$15$.

5.3.3 Simplex Method Explanation of the Method of Multipliers

The relationship between the method of multipliers and the simplex method can be explained based on the primal-dual relationships (Section 4.2). From the special structure of the LP representing the transportation model (see Example 5.1-1 for an illustration), the associated dual problem can be written as

$$\text{Maximize } z = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

subject to

$$u_i + v_j \leq c_{ij}, \text{all } i \text{ and } j$$

$$u_i \text{ and } v_j \text{ unrestricted}$$

where

a_i = Supply amount at source i

b_j = Demand amount at destination j

c_{ij} = Unit transportation cost from source i to destination j

u_i = Dual variable of the constraint associated with source i

v_j = Dual variable of the constraint associated with destination j

From Formula 2, Section 4.2.4, the objective-function coefficients (reduced costs) of the variable x_{ij} equal the difference between the left- and right-hand sides of the corresponding dual constraint—that is, $u_i + v_j - c_{ij}$. However, we know that this quantity must equal zero for each *basic variable*, which then produces the following result:

$$u_i + v_j = c_{ij}, \text{ for each } \textit{basic} \text{ variable } x_{ij}.$$

There are $m + n - 1$ such equations whose solution (after assuming an arbitrary value $u_1 = 0$) yields the multipliers u_i and v_j. Once these multipliers are computed, the entering variable is determined from all the *nonbasic* variables as the one having the largest positive $u_i + v_j - c_{ij}$.

The assignment of an arbitrary value to one of the dual variables (i.e., $u_1 = 0$) may appear inconsistent with the way the dual variables are computed using Method 2 in Section 4.2.3. Namely, for a given basic solution (and, hence, inverse), the dual values must be unique. Problem 2, Set 5.3c, addresses this point.

PROBLEM SET 5.3C

1. Write the dual problem for the LP of the transportation problem in Example 5.3-5 (Table 5.21). Compute the associated optimum *dual* objective value using the optimal dual values given in Table 5.25, and show that it equals the optimal cost given in the example.
2. In the transportation model, one of the dual variables assumes an arbitrary value. This means that for the same basic solution, the values of the associated dual variables are not unique. The result appears to contradict the theory of linear programming, where the dual values are determined as the product of the vector of the objective coefficients for the basic variables and the associated inverse basic matrix (see Method 2, Section 4.2.3). Show that for the transportation model, although the inverse basis is unique, the vector of *basic* objective coefficients need not be so. Specifically, show that if c_{ij} is changed to $c_{ij} + k$ for all i and j, where k is a constant, then the optimal values of x_{ij} will remain the same. Hence, the use of an arbitrary value for a dual variable is implicitly equivalent to assuming that a specific constant k is added to all c_{ij}.

5.4 THE ASSIGNMENT MODEL

"The best person for the job" is an apt description of the assignment model. The situation can be illustrated by the assignment of workers with varying degrees of skill to jobs. A job that happens to match a worker's skill costs less than one in which the operator is not as skillful. The objective of the model is to determine the minimum-cost assignment of workers to jobs.

The general assignment model with n workers and n jobs is represented in Table 5.31.

The element c_{ij} represents the cost of assigning worker i to job j $(i, j = 1, 2, \ldots, n)$. There is no loss of generality in assuming that the number of workers always

TABLE 5.31 Assignment Model

		Jobs				
		1	2	...	n	
Worker	1	c_{11}	c_{12}	...	c_{1n}	1
	2	c_{21}	c_{22}	...	c_{2n}	1
	⋮	⋮	⋮	⋮	⋮	⋮
	n	c_{n1}	c_{n2}	...	c_{nn}	1
		1	1	...	1	

equals the number of jobs, because we can always add fictitious workers or fictitious jobs to satisfy this assumption.

The assignment model is actually a special case of the transportation model in which the workers represent the sources, and the jobs represent the destinations. The supply (demand) amount at each source (destination) exactly equals 1. The cost of "transporting" worker i to job j is c_{ij}. In effect, the assignment model can be solved directly as a regular transportation model. Nevertheless, the fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the **Hungarian method.** Although the new solution method appears totally unrelated to the transportation model, the algorithm is actually rooted in the simplex method, just as the transportation model is.

5.4.1 The Hungarian Method[8]

We will use two examples to present the mechanics of the new algorithm. The next section provides a simplex-based explanation of the procedure.

Example 5.4-1

Joe Klyne's three children, John, Karen, and Terri, want to earn some money to take care of personal expenses during a school trip to the local zoo. Mr. Klyne has chosen three chores for his children: mowing the lawn, painting the garage door, and washing the family cars. To avoid anticipated sibling competition, he asks them to submit (secret) bids for what they feel is fair pay for each of the three chores. The understanding is that all three children will abide by their father's decision as to who gets which chore. Table 5.32 summarizes the bids received. Based on this information, how should Mr. Klyne assign the chores?

The assignment problem will be solved by the Hungarian method.

Step 1. For the original cost matrix, identify each row's minimum, and subtract it from all the entries of the row.

---

[8] As with the transportation model, the classical Hungarian method, designed primarily for *hand* computations, is something of the past and is presented here purely for historical reasons. Today, the need for such computational shortcuts is not warranted as the problem can be solved as a regular LP using highly efficient computer codes.

TABLE 5.32 Klyne's Assignment Problem

	Mow	Paint	Wash
John	\$15	\$10	\$9
Karen	\$9	\$15	\$10
Terri	\$10	\$12	\$8

Step 2. For the matrix resulting from step 1, identify each column's minimum, and subtract it from all the entries of the column.

Step 3. Identify the optimal solution as the feasible assignment associated with the zero elements of the matrix obtained in step 2.

Let p_i and q_j be the minimum costs associated with row i and column j as defined in steps 1 and 2, respectively. The row minimums of step 1 are computed from the original cost matrix as shown in Table 5.33.

Next, subtract the row minimum from each respective row to obtain the reduced matrix in Table 5.34.

The application of step 2 yields the column minimums in Table 5.34. Subtracting these values from the respective columns, we get the reduced matrix in Table 5.35.

TABLE 5.33 Step 1 of the Hungarian Method

	Mow	Paint	Wash	Row minimum
John	15	10	9	$p_1 = 9$
Karen	9	15	10	$p_2 = 9$
Terri	10	12	8	$p_3 = 8$

TABLE 5.34 Step 2 of the Hungarian Method

	Mow	Paint	Wash
John	6	1	0
Karen	0	6	1
Terri	2	4	0
Column minimum	$q_1 = 0$	$q_2 = 1$	$q_3 = 0$

TABLE 5.35 Step 3 of the Hungarian Method

	Mow	Paint	Wash
John	6	$\underline{0}$	0
Karen	$\underline{0}$	5	1
Terri	2	3	$\underline{0}$

The cells with underscored zero entries provide the optimum solution. This means that John gets to paint the garage door, Karen gets to mow the lawn, and Terri gets to wash the family cars. The total cost to Mr. Klyne is 9 + 10 + 8 = \$27. This amount also will always equal $(p_1 + p_2 + p_3) + (q_1 + q_2 + q_3) = (9 + 9 + 8)+(0 + 1 + 0) = \$27$. (A justification of this result is given in the next section.)

The given steps of the Hungarian method work well in the preceding example because the zero entries in the final matrix happen to produce a *feasible* assignment (in the sense that each child is assigned a distinct chore). In some cases, the zeros created by steps 1 and 2 may not yield a feasible solution directly, and further steps are needed to find the optimal (feasible) assignment. The following example demonstrates this situation.

Example 5.4-2

Suppose that the situation discussed in Example 5.4-1 is extended to four children and four chores. Table 5.36 summarizes the cost elements of the problem.

The application of steps 1 and 2 to the matrix in Table 5.36 (using $p_1 = 1, p_2 = 7, p_3 = 4, p_4 = 5, q_1 = 0, q_2 = 0, q_3 = 3$, and $q_4 = 0$) yields the reduced matrix in Table 5.37 (verify!).

The locations of the zero entries do not allow assigning unique chores to all the children. For example, if we assign child 1 to chore 1, then column 1 will be eliminated, and child 3 will not have a zero entry in the remaining three columns. This obstacle can be accounted for by adding the following step to the procedure outlined in Example 5.4-1:

Step 2a. If no feasible assignment (with all zero entries) can be secured from steps 1 and 2,

(i) Draw the *minimum* number of horizontal and vertical lines in the last reduced matrix that will cover *all* the zero entries.

TABLE 5.36 Assignment Model

		Chore			
		1	2	3	4
Child	1	\$1	\$4	\$6	\$3
	2	\$9	\$7	\$10	\$9
	3	\$4	\$5	\$11	\$7
	4	\$8	\$7	\$8	\$5

TABLE 5.37 Reduced Assignment Matrix

		Chore			
		1	2	3	4
Child	1	**0**	3	2	2
	2	2	**0**	**0**	2
	3	**0**	1	4	3
	4	3	2	**0**	**0**

TABLE 5.38 Application of Step 2a

		Chore			
		1	2	3	4
Child	1	0	3	2	2
	2	**2**	0	0	2
	3	0	*1*	4	3
	4	**3**	2	0	0

TABLE 5.39 Optimal Assignment

		Chore			
		1	2	3	4
Child	1	$\underline{0}$	2	1	1
	2	3	0	$\underline{0}$	2
	3	0	$\underline{0}$	3	2
	4	4	2	0	$\underline{0}$

(ii) Select the *smallest un*covered entry, subtract it from every uncovered entry, then add it to every entry at the intersection of two lines.

(iii) If no feasible assignment can be found among the resulting zero entries, repeat step 2a. Otherwise, go to step 3 to determine the optimal assignment.

The application of step 2a to the last matrix produces the shaded cells in Table 5.38. The smallest unshaded entry (shown in italics) equals 1. This entry is added to the bold intersection cells and subtracted from the remaining shaded cells to produce the matrix in Table 5.39.

The optimum solution (shown by the underscored zeros) calls for assigning child 1 to chore 1, child 2 to chore 3, child 3 to chore 2, and child 4 to chore 4. The associated optimal cost is $1 + 10 + 5 + 5 = \$21$. The same cost is also determined by summing the p_i's, the q_j's, and the entry that was subtracted after the shaded cells were determined—that is, $(1 + 7 + 4 + 5) + (0 + 0 + 3 + 0) + (1) = \$21$.

AMPL Moment.

File amplEx5.4-2.txt provides the AMPL model for the assignment model. The model is very similar to that of the transportation model.

PROBLEM SET 5.4A

1. Solve the assignment models in Table 5.40.
   - **(a)** Solve by the Hungarian method.
   - **(b)** *TORA Experiment.* Express the problem as an LP and solve it with TORA.
   - **(c)** *TORA Experiment.* Use TORA to solve the problem as a transportation model.

TABLE 5.40 Data for Problem 1

(i)					(ii)				
$3	$8	$2	$10	$3	$3	$9	$2	$3	$7
$8	$7	$2	$9	$7	$6	$1	$5	$6	$6
$6	$4	$2	$7	$5	$9	$4	$7	$10	$3
$8	$4	$2	$3	$5	$2	$5	$4	$2	$1
$9	$10	$6	$9	$10	$9	$6	$2	$4	$5

(d) *Solver Experiment.* Modify Excel file solverEx5.3-1.xls to solve the problem.

(e) *AMPL Experiment.* Modify amplEx5.3-1b.txt to solve the problem.

2. JoShop needs to assign 4 jobs to 4 workers. The cost of performing a job is a function of the skills of the workers. Table 5.41 summarizes the cost of the assignments. Worker 1 cannot do job 3 and worker 3 cannot do job 4. Determine the optimal assignment using the Hungarian method.

3. In the JoShop model of Problem 2, suppose that an additional (fifth) worker becomes available for performing the four jobs at the respective costs of $60, $45, $30, and $80. Is it economical to replace one of the current four workers with the new one?

4. In the model of Problem 2, suppose that JoShop has just received a fifth job and that the respective costs of performing it by the four current workers are $20, $10, $20, and $80. Should the new job take priority over any of the four jobs JoShop already has?

***5.** A business executive must make the four round trips listed in Table 5.42 between the head office in Dallas and a branch office in Atlanta.

The price of a round-trip ticket from Dallas is $400. A discount of 25% is granted if the dates of arrival and departure of a ticket span a weekend (Saturday and Sunday). If the stay in Atlanta lasts more than 21 days, the discount is increased to 30%. A one-way

TABLE 5.41 Data for Problem 2

		Job			
		1	2	3	4
Worker	1	$50	$50	—	$20
	2	$70	$40	$20	$30
	3	$90	$30	$50	—
	4	$70	$20	$60	$70

TABLE 5.42 Data for Problem 5

Departure date from Dallas	Return date to Dallas
Monday, June 3	Friday, June 7
Monday, June 10	Wednesday, June 12
Monday, June 17	Friday, June 21
Tuesday, June 25	Friday, June 28

ticket between Dallas and Atlanta (either direction) costs $250. How should the executive purchase the tickets?

***6.** Figure 5.6 gives a schematic layout of a machine shop with its existing work centers designated by squares 1, 2, 3, and 4. Four new work centers, I, II, III, and IV, are to be added to the shop at the locations designated by circles *a*, *b*, *c*, and *d*. The objective is to assign the new centers to the proposed locations to minimize the total materials handling traffic between the existing centers and the proposed ones. Table 5.43 summarizes the frequency of trips between the new centers and the old ones. Materials handling equipment travels along the rectangular aisles intersecting at the locations of the centers. For example, the one-way travel distance (in meters) between center 1 and location *b* is 30 + 20 = 50 m.

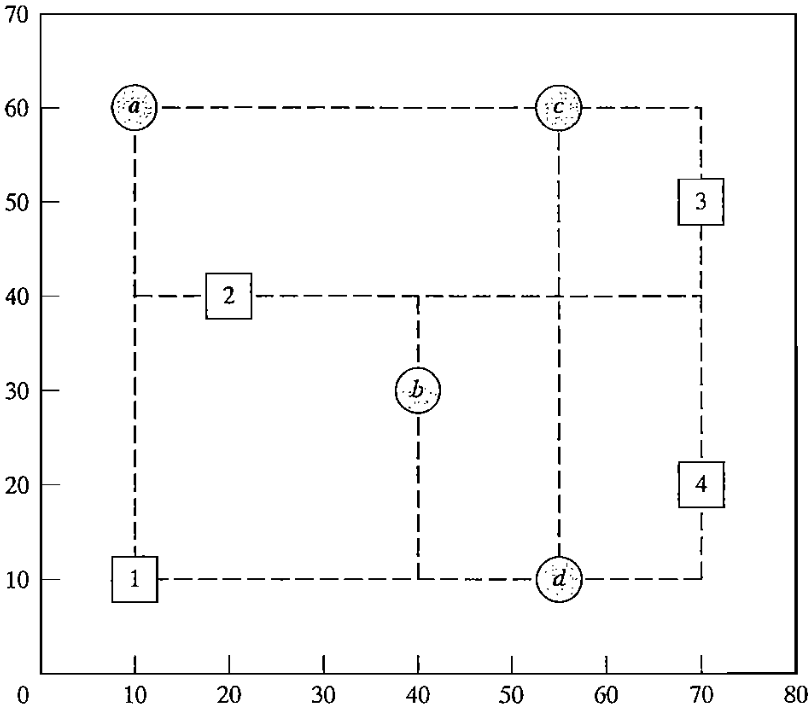


FIGURE 5.6

Machine shop layout for Problem 6, Set 5.4a

TABLE 5.43 Data for Problem 6

	New center			
	I	II	III	IV
Existing center 1	10	2	4	3
2	7	1	9	5
3	0	8	6	2
4	11	4	0	7

7. In the Industrial Engineering Department at the University of Arkansas, INEG 4904 is a capstone design course intended to allow teams of students to apply the knowledge and skills learned in the undergraduate curriculum to a practical problem. The members of each team select a project manager, identify an appropriate scope for their project, write and present a proposal, perform necessary tasks for meeting the project objectives, and write and present a final report. The course instructor identifies potential projects and provides appropriate information sheets for each, including contact at the sponsoring organization, project summary, and potential skills needed to complete the project. Each design team is required to submit a report justifying the selection of team members and the team manager. The report also provides a ranking for each project in order of preference, including justification regarding proper matching of the team's skills with the project objectives. In a specific semester, the following projects were identified: Boeing F-15, Boeing F-18, Boeing Simulation, Cargil, Cobb-Vantress, ConAgra, Cooper, DaySpring (layout), DaySpring (material handling), J.B. Hunt, Raytheon, Tyson South, Tyson East, Wal-Mart, and Yellow Transportation. The projects for Boeing and Raytheon require U.S. citizenship of all team members. Of the eleven design teams available for this semester, four do not meet this requirement.

   Devise a procedure for assigning projects to teams and justify the arguments you use to reach a decision.

5.4.2 Simplex Explanation of the Hungarian Method

The assignment problem in which n workers are assigned to n jobs can be represented as an LP model in the following manner: Let c_{ij} be the cost of assigning worker i to job j, and define

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ is assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Then the LP model is given as

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n$$

$$x_{ij} = 0 \text{ or } 1$$

The optimal solution of the preceding LP model remains unchanged if a constant is added to or subtracted from any row or column of the cost matrix (c_{ij}). To prove this point, let p_i and q_j be constants subtracted from row i and column j. Thus, the cost element c_{ij} is changed to

$$c'_{ij} = c_{ij} - p_i - q_j$$

5.5

Now

$$\sum_i \sum_j c'_{ij}x_{ij} = \sum_i \sum_j (c_{ij} - p_i - q_j)x_{ij} = \sum_i \sum_j c_{ij}x_{ij} - \sum_i p_i\left(\sum_j x_{ij}\right) - \sum_j q_j\left(\sum_i x_{ij}\right)$$

$$= \sum_i \sum_j c_{ij}x_{ij} - \sum_i p_i(1) - \sum_j q_j(1)$$

$$= \sum_i \sum_j c_{ij}x_{ij} - \text{constant}$$

Because the new objective function differs from the original one by a constant, the optimum values of x_{ij} must be the same in both cases. The development thus shows that steps 1 and 2 of the Hungarian method, which call for subtracting p_i from row i and then subtracting q_j from column j, produce an equivalent assignment model. In this regard, if a feasible solution can be found among the zero entries of the cost matrix created by steps 1 and 2, then it must be optimum because the cost in the modified matrix cannot be less than zero.

If the created zero entries cannot yield a feasible solution (as Example 5.4-2 demonstrates), then step 2a (dealing with the covering of the zero entries) must be applied. The validity of this procedure is again rooted in the simplex method of linear programming and can be explained by duality theory (Chapter 4) and the complementary slackness theorem (Chapter 7). We will not present the details of the proof here because they are somewhat involved.

The reason $(p_1 + p_2 + \cdots + p_n) + (q_1 + q_2 + \cdots + q_n)$ gives the optimal objective value is that it represents the dual objective function of the assignment model. This result can be seen through comparison with the dual objective function of the transportation model given in Section 5.3.4. [See Bazaraa and Associates (1990, pp. 499–508) for the details.]

5.5 THE TRANSSHIPMENT MODEL

The transshipment model recognizes that it may be cheaper to ship through intermediate or *transient* nodes before reaching the final destination. This concept is more general than that of the regular transportation model, where direct shipments only are allowed between a source and a destination.

This section shows how a transshipment model can be converted to (and solved as) a regular transportation model using the idea of a **buffer**.

Example 5.5-1

Two automobile plants, *P*1 and *P*2, are linked to three dealers, *D*1, *D*2, and *D*3, by way of two transit centers, *T*1 and *T*2, according to the network shown in Figure 5.7. The supply amounts at plants *P*1 and *P*2 are 1000 and 1200 cars, and the demand amounts at dealers *D*1, *D*2, and *D*3, are 800, 900, and 500 cars. The shipping costs per car (in hundreds of dollars) between pairs of nodes are shown on the connecting links (or arcs) of the network.

Transshipment occurs in the network in Figure 5.7 because the entire supply amount of 2200 (= 1000 + 1200) cars at nodes *P*1 and *P*2 could conceivably pass through any node of the

Set 4.4a

1. (b) No, because point E is feasible and the dual simplex must stay infeasible until optimum is reached.

4. (c) Add the artificial constraint $x_1 \leq M$. Problem has no feasible solution.

Set 4.5a

4. Let Q be the weekly feed in lb (= 5200, 9600, 15000, 20000, 26000, 32000, 38000, 42000, for weeks 1, 2, ..., and 8). Optimum solution: Limestone = .028Q, corn = .649Q, and soybean meal = .323Q. Cost = .81221Q.

Set 4.5b

1. (a) Additional constraint is redundant.

Set 4.5c

2. (a) New dual values = $\left(\frac{1}{2}, 0, 0, 0\right)$. Current solution remains optimal.

(c) New dual values = $\left(-\frac{1}{8}, \frac{11}{4}, 0, 0\right)$. $z - .125s_1 + 2.75s_2 = 13.5$. New solution: $x_1 = 2, x_2 = 2, x_3 = 4, z = 14$.

Set 4.5d

1. $\frac{p}{100}(y_1 + 3y_2 + y_3) - 3 \geq 0$. For $y_1 = 1$, $y_2 = 2$, and $y_3 = 0$, $p \geq 42.86\%$.

3. (a) Reduced cost for fire engines $= 3y_1 + 2y_2 + 4y_3 - 5 = 2 > 0$. Fire engines are not profitable.

CHAPTER 5

Set 5.1a

4. Assign a very high cost, M, to the route from Detroit to dummy destination.

6. (a and b) Use $M = 10{,}000$. Solution is shown in bold. Total cost = \$49,710.

	1	2	3	Supply
Plant 1	600	700	400 **25**	**25**
Plant 2	320 **23**	300 **17**	350	**40**
Plant 3	500	480 **25**	450 **5**	**30**
Excess Plant 4	1000 **13**	1000	M	**13**
Demand	**36**	**42**	**30**	

(c) City 1 excess cost = \$13,000.

9. Solution (in million gallons) is shown in bold. Area 2 will be 2 million gallons short. Total cost = $304,000.

	A1	A2	A3	Supply
Refinery 1	12 **4**	18 **2**	M	**6**
Refinery 2	30	10 **4**	8 **1**	**5**
Refinery 3	20	25	12 **6**	**6**
Dummy	M	50 **2**	50	**2**
Demand	**4**	**8**	**7**	

Set 5.2a

2. Total cost = $804. Problem has alternative optima.

		Sharpening service			
Day	*New*	*Overnight*	*2-day*	*3-day*	Disposal
Monday	24	0	6	18	0
Tuesday	12	12	0	0	0
Wednesday	2	14	0	0	0
Thursday	0	0	20	0	0
Friday	0	14	0	0	4
Saturday	0	2	0	0	12
Sunday	0	0	0	0	22

5. Total cost = $190,040. Problem has alternative optima.

Period	Capacity	Produced amount	Delivery
1	500	500	400 for (period) 1 and 100 for 2
2	600	600	200 for 2, 220 for 3, and 180 for 4
3	200	200	200 for 3
4	300	200	200 for 4

Set 5.3a

1. (a) Northwest: cost = $42. Least-cost: cost = $37. Vogel: cost = $37.

Set 5.3b

5. (a) Cost = $1475.
   (b) $c_{12} \geq 3, c_{13} \geq 8, c_{23} \geq 13, c_{31} \geq 7$.