$u(x,t) = \underline{X(x)\ T(t)}$

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Applied Maths Series

Introduction → In 19th century Sophus Lie introduce the notion of continuous group, now known as Lie groups.

Book → Similarity method for D.E. (1974) G.W. Bluman & J.D. Cole
(1) Symmetries & D.E. George Bluman & Kumar
(2) P.D.E by Dr. chris Tisdell

5.8 Similarity Solutions to PDE

View this lesson on YouTube [30]

Where are we going?

- We will discover a new method of solving PDE, known as the "similarity solution" approach. or "similarity method."
- In contrast to the method of "separation of variables", the new method actually involves connecting or combining the variables in a special way.
- Such techniques are very powerful, enabling us to solve a range of linear and nonlinear problems.
- The basic idea is to determine a set of "invariant stretching transformations" that leaves the PDE in question unchanged. In this way we can reduce the problem to an ODE which may be solvable.

A very famous method "Seperation of Variables" actually seperates the independent variables of solution. But "Similarity method" is reverse of "SOV method". It actually connect or combine the independent variables, in a special way rather than ~~rep~~ ~~sep~~ ~~repleted~~ ~~repleting~~ reperating the variables. This method (S.M.) is very powerful because it solves both linear and non-linear Problems.

If we have a first order D.E., If the general eq^n is

$F(x, y, \frac{dy}{dx}) = 0$ --- (1)

the special case is

$F(x, \frac{dy}{dx}) = 0$ --- (2)

Now (2) can be written as $\frac{dy}{dx} = G(x)$ --- (3)

& $y = \int_{x_0}^{x} G(u)\,du + \alpha$ constant --- (4)

is general solution of (3)

We have already seen the idea of "invariant stretching transformations" at work when we solved the heat/diffusion equation

$$u_t = ku_{xx} \qquad (5.8.1)$$

where $k > 0$ is a constant. If we let

$$X := \sqrt{a}x, \quad \text{and } T := at$$

then $u(X, T)$ solves (5.8.1) and notice that the ratio

$$\frac{X^2}{T} = \frac{ax^2}{at} = \frac{x^2}{t}$$

or

$$\frac{X}{\sqrt{T}} = \frac{\sqrt{a}x}{\sqrt{at}} = \frac{x}{\sqrt{t}}.$$

This kinds of transformations are known as *invariant* with respect to the PDE. (5.8.1).

Based on the above, we assumed solutions to (5.8.1) were of the form like

$$u(x, t) = g\left(\frac{x}{\sqrt{4kt}}\right)$$

and then (5.8.1) reduced to an ODE

$$g'' + 2pg' = 0, \quad p = x/\sqrt{4kt} \quad (5.8.2)$$

which we could solve to obtain

$$u(x, t) = g(p) = A\int_0^{x/\sqrt{4kt}} e^{-s^2}\, ds + B.$$

We now show how to go further!

Consider, (say) the PDE

$$L(u) = 0.$$

Consider the transformations

$$X = a^\alpha x, \quad Y = a^\beta y, \quad \bar{u} = a^\gamma u.$$

From the PDE under consideration we wish to determine values for α, β, γ such that

$$L(\bar{u}) = 0.$$

Then construct solutions of form

$$u(x, y) = y^{-r}g(xy^{-s})$$

for some r and s.

Form and solve an appropriate ordinary differential equation.

The procedure is best explained via examples.

Consider $\frac{dy}{dx} = F(x,y)$ -- -(1)

Assume the D.E. is invariant under special transformation known as "streching transformation"

$$\left.\begin{array}{l} x^* = \alpha x \\ y^* = \beta y \end{array}\right\} \quad 0<\alpha,\beta<\infty \quad \text{-- (2)}$$

A direction field at P^* (image space) is also assigned by the transformation of the D.E.

$$\frac{dy^*}{dx^*} = \frac{\beta}{\alpha}F\left(\frac{x^*}{\alpha}, \frac{y^*}{\beta}\right) \text{-- (3)}$$

Now, the D.E. is s.t.b. invariant under the transformation (2) when the ~~(5.8.2)~~ D.E reads the same in the new coordinates

$$\boxed{\frac{\beta}{\alpha}F\left(\frac{x^*}{\alpha}, \frac{y^*}{\beta}\right) = F(x^*, y^*)}$$

$$\frac{dy^*}{dx^*} = \frac{\beta}{\alpha}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\alpha}{\beta}\frac{dy^*}{dx^*}$$

$$\frac{dy}{dx} = F\left(\frac{x^*}{\alpha}, \frac{y^*}{\beta}\right)$$

$$\Rightarrow F\left(\frac{x^*}{\alpha}, \frac{y^*}{\beta}\right) = \frac{\alpha}{\beta}\frac{dy^*}{dx^*}$$

$$\Rightarrow \boxed{\frac{dy^*}{dx^*} = \frac{\beta}{\alpha}F\left(\frac{x^*}{\alpha}, \frac{y^*}{\beta}\right)}$$

Example.

Solve, by applying the similarity solution method:

$$u_t = ku_{xx};$$ → Heat Eq^n / diffusion eq^n. ; $k>0$ positive constant

$$u(x,0)=0,\quad x>0;$$

$$u(x,t)\to 0,\quad x\to\infty;\quad u_x(0,t)=N,\quad t>0.$$

where u might represent the temp. ~~for a semi-infinite~~ at position x and time t.

Solution: #1. Determine a set of stretching (dilation) transformations under which the PDE is invariant (unchanged).
Consider

$$\boxed{\bar{X}=a^{\alpha}x;\quad \bar{T}=a^{\beta}t;\quad \bar{u}(\bar{X},\bar{T})=a^{\gamma}u(x,t).}$$

Now, by the chain rule we have $x = a^{-\alpha}\bar{X}$, $t = a^{-\beta}\bar{T}$, $u = a^{-\gamma}\bar{u}(\bar{X},\bar{T})$

$$\bar{u}_{\bar{T}} = a^{\gamma}u_t t_{\bar{T}} = a^{\gamma-\beta}u_t.$$

$\bar{u}_{\bar{T}} = \bar{u}_t\, t_{\bar{T}} = a^{\gamma}u_t\, a^{-\beta}(1) = a^{\gamma-\beta}u_t$

Similarly, $\bar{u}_{\bar{X}} = a^{\gamma}u_t x_{\bar{X}} = a^{\gamma-\alpha}u_x$ and $\bar{u}_{\bar{X}\bar{X}} = a^{\gamma-\alpha}(u_x)_{\bar{X}} = a^{\gamma-2\alpha}u_{xx}$. Hence

$\bar{u}_x x_{\bar{X}} = a^{\gamma}u_x a^{-\alpha}(1) = a^{\gamma-\alpha}u_x$

$$\begin{aligned}\bar{u}_{\bar{T}} - k\bar{u}_{\bar{X}\bar{X}} &= a^{\gamma-\beta}u_t - ka^{\gamma-2\alpha}u_{xx}\\ &= 0\end{aligned}$$

$\bar{u}_{\bar{X}\bar{X}} = (\bar{u}_{\bar{X}})_{\bar{X}} = (\bar{u}_{\bar{X}})_x\, x_{\bar{X}} = (a^{\gamma-\alpha})u_{xx}\, a^{-\alpha}(1) = a^{\gamma-2\alpha}u_{xx}$

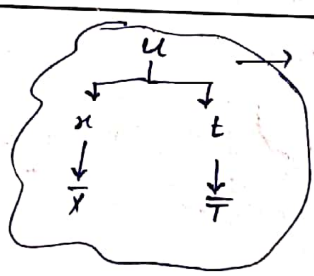
when $\gamma-\beta=\gamma-2\alpha$. That is, our PDE is invariant under the stretching transformations when

$$\beta = 2\alpha$$

and γ is arbitrary. This means that when $u=u(x,t)$ solves the PDE then so does $a^{\gamma}u(a^{\alpha}x, a^{\beta}t)$.

$\bar{u}(\bar{X},\bar{T}) = \bar{u}(\bar{X},\bar{T})$

$\bar{u}_{\bar{T}} = a^{\gamma}u_{\bar{T}}$ ~~$= a^{\gamma}u_t$~~



u depends on x & t & x depends on $\bar{X}$ & t depends on $\bar{T}$.

(i) Find $\bar{u}_{\bar{T}}(\bar{X},\bar{T}) = \bar{u}_t\, t_{\bar{T}}$

$= a^{\gamma}u_t\, a^{-\beta}(1)$

$$\boxed{\bar{u}_{\bar{T}}(\bar{X},\bar{T}) = a^{\gamma-\beta}u_t}$$

(ii) $\bar{u}_{\bar{X}}(\bar{X},\bar{T}) = \bar{u}_x\, x_{\bar{X}} = a^{\gamma}u_x\, a^{-\alpha}(1)$

$= a^{\gamma-\alpha}u_x$

$$\boxed{\bar{u}_{\bar{X}\bar{X}}(\bar{X},\bar{T}) = (\bar{u}_{\bar{X}})_x\, x_{\bar{X}} = a^{\gamma-\alpha}u_{xx}\, a^{-\alpha} = a^{\gamma-2\alpha}u_{xx}}$$

Example.

#2. Determine s and r such that $\bar{X}\bar{T}^s = xt^s$ and $\bar{u}\bar{T}^r = ut^r$. We have

$$\begin{aligned}\bar{X}\bar{T}^s &= a^\alpha x(a^\beta t)^s = a^\alpha x(a^{2\alpha}t)^s \qquad (\because \beta = 2\alpha)\\ &= a^{\alpha(1+2s)}xt^s\\ &= xt^s\end{aligned}$$

for $s = -1/2$. Hence $xt^{-1/2}$ forms an absolute invariant and we let $p = xt^{-1/2}$. Now

$$\begin{aligned}\bar{u}\bar{T}^r &= a^\gamma \bar{u}(a^\beta t)^r = a^\gamma u(a^{2\alpha}t)^r\\ &= a^{\gamma+2\alpha r}ut^r \qquad (\because \beta = 2\alpha)\\ &= ut^r\end{aligned}$$

for $r = -\gamma/2\alpha$.

#3. Form $u(x,t) = t^{-r}g(xt^s) = t^{-r}g(p)$.

For our situation, we have

$$u(x,t) = t^{\gamma/2\alpha}g(xt^{-1/2}) = t^{\gamma/2\alpha}g(p).$$

We can determine γ and α from our boundary conditions. Observe

$$\begin{aligned}u_x(0,t) &= t^{\gamma/2\alpha-1/2}g'(xt^{-1/2})]_{x=0} = t^{\left(\frac{\gamma}{2\alpha}-\frac{1}{2}\right)} g'(xt^{-1/2})\\ &= t^{\gamma/2\alpha-1/2}g'(0) = t^{\left(\frac{\gamma}{2\alpha}-\frac{1}{2}\right)} g'(0)\\ &= N\end{aligned}$$

if $\gamma/2\alpha = 1/2$. Thus, $u(x,t) = t^{1/2}g(xt^{-1/2}) = t^{1/2}g(p)$. Furthermore, we have formed the boundary condition $g'(0) = N$. Now,

$$\frac{x}{t^{1/2}}$$

$$\mathrm{I} = f, \ \mathrm{II} = t^{1/2}$$

$$\begin{aligned}u(x,t) &= t^{1/2}g(xt^{-1/2})\\ &\to 0\end{aligned}$$

as $x \to \infty$ leads us to $g(p) \to 0$ as $p \to \infty$.

All we need now is an ODE in g. We compute the derivatives that appear in our original PDE, namely

$$\begin{aligned}u_t &= \left[t^{1/2}g(xt^{-1/2})\right]_t\\ &= \frac{1}{2}t^{-1/2}g + t^{1/2}\left(-\frac{x}{2t^{3/2}}\right)g'\\ &= \frac{t^{-1/2}}{2}\left[g - pg'\right].\end{aligned}$$

$$\begin{aligned}u_t &= \left[t^{1/2}g(xt^{-1/2})\right]_t\\ &= \tfrac{1}{2}t^{-1/2}g + t^{1/2}g'(xt^{-1/2})\,x\left(-\tfrac{1}{2}t^{-3/2}\right)\\ &= \frac{t^{-1/2}}{2}\left(g - g'p\right)\end{aligned}$$

Also,

$$u_{xx} = \left[t^{1/2}g(xt^{-1/2})\right]_{xx} = t^{-1/2}g''.$$

$$u_x = t^{1/2}g'(xt^{-1/2})\,t^{-1/2}$$

$$u_{xx} = t^{1/2}g''(xt^{-1/2})\,t^{-1} = t^{-1/2}g''$$

Example.

Thus, our PDE becomes

$$\begin{aligned} 0 &= u_t - k u_{xx} \\ &= \frac{t^{-1/2}}{2}[g - pg'] - kt^{-1/2}g'' \end{aligned}$$

that, is

$$2kg'' + pg' - g = 0. \qquad (5.8.3)$$

→ 2^{nd} order homo linear Ordinary DE. where k is constant and p is the independent variable of the fun g.

Now, we notice that one set of solutions to this ODE is $g_1(p) = C_1 p$, where C_1 is a constant. (∵ $g'(p) = C(1)$, $g''(p) = 0$, $2k(0) + pC - Cp = 0$)

We may construct a second solution via the reduction of order method, via

$$g_2(p) = ph(p)$$

where h (→ unknown fun) is a function to be determined. Calculate g_2', g_2'' etc and substitute into (5.8.3). Now, our ODE (5.8.3) becomes

$$\boxed{2kph'' + (4k + p^2)h' = 0}$$

$g(p) = ph(p)$
$g'(p) = h(p) + ph'(p)$
$g''(p) = h'(p) + h'(p) + ph''(p)$

$2k(ph''(p) + 2h'(p)) + p[h(p) + ph'(p)] - ph(p) = 0$

→ $\boxed{2kph'' + (4k + p^2)h' = 0}$

so that (Integration)

$$\begin{aligned} h'(p) &= C_2 e^{-\int (4k+p^2)/2kp \, dp} \\ &= C_2 e^{-\int (2/p + p/2k) \, dp} \\ &= C_2 e^{-(2\ln p + p^2/4k)} \\ &= \frac{C_2}{p^2} e^{-p^2/4k}. \end{aligned}$$

(where C_2 is arb: constant.)

Hence

$$h(\infty) - h(p) = \int_p^\infty \frac{C_2}{s^2} e^{-s^2/4k} \, ds$$

($\lim_{p\to\infty} h(p) = M$)

and denote $M := h(\infty)$. Now, integrating by parts we have

$$h(p) = M - C_2 \left(\frac{1}{p} e^{-p^2/4k} - \frac{1}{2k} \int_p^\infty e^{-s^2/4k} \, ds \right).$$

$h(p) = M - C_2 \left[\int_p^\infty \frac{e^{-s^2/4k}}{s^2} ds \right]$

$= M - C_2 \left[\left(e^{-s^2/4k} \left(\frac{-1}{s}\right) \right)_p^\infty - \int_p^\infty \left(\frac{-1}{s}\right) e^{-s^2/4k} \left(\frac{-2s}{4k}\right) ds \right]$

$= M - C_2 \left[e^{-p^2/4k} \left(\frac{1}{p}\right) - \frac{1}{2k} \int_p^\infty e^{-s^2/4k} ds \right]$

Thus, ~~$g(p) = g_1(p) + g_2(p)$~~ $g(p) = g_1(p) + g_2(p)$

$$g(p) = (M + C_1)p - C_2 \left(e^{-p^2/4k} - \frac{p}{2k} \int_p^\infty e^{-s^2/4k} \, ds \right).$$

Applying our boundary conditions we see that $g(\infty) = 0$ for $M + C_1 = 0$, hence

$$\boxed{g(p) = -C_2 \left(e^{-p^2/4k} - \frac{p}{2k} \int_p^\infty e^{-s^2/4k} \, ds \right).}$$

$C_1 p + p\left[M - C_2 \left(\frac{1}{p} e^{-p^2/4k} - \frac{1}{2k} \int_p^\infty e^{-s^2/4k} ds \right) \right]$

$= C_1 p + Mp - C_2 \left(e^{-p^2/4k} - \frac{p}{2k} \int_p^\infty e^{-s^2/4k} ds \right)$

$= (C_1 + M)p - C_2 \left(e^{-p^2/4k} - \frac{p}{2k} \int_p^\infty e^{-s^2/4k} ds \right)$

$$g'(p) = -C_2\left(e^{-p^2/4k}\left(\frac{-2p}{4k}\right) - \frac{1}{2k}\int_p^\infty e^{-s^2/4k}\,ds\right)$$

$$g'(0) = +\frac{C_2}{2k}\left(0 + \int_0^\infty e^{-s^2/4k}\,ds\right) = \frac{C_2}{2k}\left(\int_0^\infty e^{-s^2/4k}\,ds\right)$$

Example.

In addition

$$N = g'(0) = \frac{C_2}{2k}\int_0^\infty e^{-s^2/4k}\,ds.$$

$s^2/4k = z^2$

If we let $z = s/2\sqrt{k}$ then $dz = ds/2\sqrt{k}$ and so

$$\begin{aligned} N &= \frac{C_2}{\sqrt{k}}\int_0^\infty e^{-z^2}\,dz \\ &= \frac{C_2}{\sqrt{k}}\frac{\sqrt{\pi}}{2}. \end{aligned}$$

Hence

$$C_2 = \frac{2N\sqrt{k}}{\sqrt{\pi}}.$$

Thus

$$g(p) = -\frac{2N\sqrt{k}}{\sqrt{\pi}}\left[e^{-p^2/4k} - \frac{p}{2k}\int_p^\infty e^{-s^2/4k}\,ds\right]$$

#4. We have solved our ODE!! Finally, we have

$$\begin{aligned} u(x,t) &= t^{1/2}g(xt^{-1/2}) \\ &= -t^{1/2}\frac{2N\sqrt{k}}{\sqrt{\pi}}\left[e^{-x^2/4kt} - \frac{x}{2k\sqrt{t}}\int_{x/\sqrt{t}}^\infty e^{-x^2/4kt}\,ds\right]. \end{aligned}$$

If desired, this can be written in terms of a complementary error function.

Gamma function ⇒

$$\Gamma(x) = \int_0^\infty e^{-t}\,t^{x-1}\,dt \qquad x>0$$

$\Gamma(1) = 1$, $\Gamma(2) = 1$

$\Gamma(3) = 2\Gamma(2) = 2\times1 = 2!$

$\Gamma(4) = 3\Gamma(3) = 3\times2\times1 = 6 = 3!$

$\Gamma(n+1) = n!$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$z^2 = t$

$2z\,dz = dt$

$dz = \frac{1}{2z}\,dt$

$$\frac{C_2}{\sqrt{k}}\int_0^\infty \frac{e^{-t}}{2z}\,dt = \frac{C_2}{\sqrt{k}}\,\frac{1}{2}\int_0^\infty z^{-1}e^{-t}\,dt$$

Similarity solution method summary

1. Determine a set of stretching transformations

$$\bar{X} = a^\alpha x;\quad \bar{T} = a^\beta t;\quad \bar{u} = a^\gamma u$$

under which the PDE is invariant.

2. Determine s and r such that

$$\boxed{\bar{X}\bar{T}^s = xt^s, \quad \bar{u}\bar{T}^r = ut^r.}$$

3. Form

$$u(x,t) = t^{-r}g(xt^s) = t^{-r}g(p)$$

and an appropriate ODE / boundary value problem for g in p.

4. Solve (if possible).