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Problem 4.2 pg No - 184 Cheney

Q4 Prove that there is no contraction of X on to X. If X is compact metric space having at least two points.

Sol^n Since (X,d) be a compact metric space.

and $d: X\times X \to \mathbb{R}$ is continuous map.

As $X\times X$ is compact
d is cts.

$d(X\times X)$ is compact hence closed and bounded, hence attains maximum value.

thus, $\exists\ a,b \in X$ such that

$$\max_{x,y\in X} d(x,y) = d(a,b).$$

Claim there is no contraction of X on-to X.

Let if possible, F be a on-to contraction map

i.e, $F: X \to X$ contraction on to map

$\exists\ a', b' \in X$ such that

$F(a') = a$, $F(b') = b$.

$$d(a,b) = d(F(a'), F(b')) \leq \theta\, d(a,b) < d(a,b)$$

$\theta \in [0,1)$

$\rightarrow\leftarrow$

where metric on $X\times X$ is

$$d_1[(x_1,y_1),(x_2,y_2)] = d(x_1,x_2) + d(y_1,y_2)$$

Continuity of metric space: $f: X\times X \to \mathbb{R}$
$f(x,y) = d(x,y)$

T.S f is cts.

Let $\{(x_n,y_n)\}$ be seq in $X\times X$ s.t. $(x_n,y_n)\to(x,y)$
i.e, $x_n \to x$
$y_n \to y$

i.e., $\left.\begin{array}{l} x_n \to x \\ y_n \to y \end{array}\right\} \Rightarrow \begin{array}{l} d(x_n, x) \to 0 \\ d(y_n, y) \to 0 \end{array}$ as $n \to \infty$.

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$$
$$\leq \underbrace{d(x_n, x)}_{\to 0} + d(x, y) + \underbrace{d(y, y_n)}_{\to 0}$$

$\underline{d(x_n, y_n) - d(x, y) \leq \varepsilon} \quad \forall \varepsilon > 0$

$$d(x, y) \leq d(x_n, x) + d(x_n, y)$$
$$\leq d(x_n, x) + d(y_n, y) + d(x_n, y_n)$$

$\underline{d(x, y) - d(x_n, y_n) \leq \varepsilon}$

we have that

$$|d(x_n, y_n) - d(x, y)| \leq \varepsilon$$

hence d is cts.

Q5 Given an example of a ~~compact~~ Complete space X and a map $F: X \to X$ having both of the following properties

a) $\|Fx - Fy\| < \|x - y\|$ whenever $x \neq y$

(b) $Fx \neq x \quad \forall x \in X$.

Sol^n consider $X = [1, \infty)$

$F(x) = x + \frac{1}{x}$ where $F: X \to X$ mapping

clearly $Fx \neq x \quad \forall x \in [1, \infty)$.

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T.S

$|Fx - Fy| < |x-y|$

$$\left|x+\frac{1}{x}-y-\frac{1}{y}\right| = \left|x-y+\frac{y-x}{xy}\right|$$

$$= \left|(x-y)\left[1-\frac{1}{xy}\right]\right| \qquad xy-1 < xy$$

$$< \left|(x-y)\frac{\cancel{xy}}{\cancel{xy}}\right|$$

$$\Rightarrow \left|x+\frac{1}{x}-y-\frac{1}{y}\right| < |x-y|$$

$|Fx - Fy| < |x-y|$ hence it is follows.

Q 7. Prove that if X is a compact metric space and F is a mapping from X to X such that $d(Fx, Fy) < d(x,y)$ when $x \neq y$ then F has unique fixed point.

Sol: Given $(X; d)$ compact metric space

$F: X \to X$ mapping

s.that $d(Fx, Fy) < d(x,y)$ $x \neq y$

T.S F has unique fixed point

Define $g(x) = d(Fx, x)$

hence g is a cts map being composition of (Fx, x) and d.

$F: X \to X$ cts

$x: X \to X$ identity.

$(Fx, x): X \times X \to X \times X$ cts

$d: X \times X \xrightarrow{cts} \mathbb{R}$

As $X \times X$ is compact

and g is cts. $[g : X \xrightarrow{cts} \mathbb{R}]$

hence g attains infimum on X.

$\exists$ $x_0 \in X$ such that

$$g(x_0) = \inf_{x \in X} d(Fx, x) = d(Fx_0, x_0).$$

Case (i)

if $g(x_0) = 0$

$\Rightarrow$ $d(Fx_0, x_0) = 0$

$Fx_0 = x_0$

hence x_0 is the fixed point

Case (ii)

if $g(x_0) > 0$

then $\underline{d(F^2 x_0, Fx_0)} < d(Fx_0, x_0) = \underbrace{g(x_0)}_{\text{infimum}}$

$\rightarrow\leftarrow$

Uniqueness of fixed point:

Let u, v be two fixed point of F.
$(u \neq v)$

$d(u,v) = d(Fu, Fv) < d(u,v)$ $\rightarrow\leftarrow$

as $Fu = u$ being fixed point, hence the proof follows
$Fv = v$