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Problem 4.2

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Q4 prove that there is no contraction of X on to X , if X is compact metric space having at least two points.

Solⁿ Since (X, d) be a compact metric space.

and $d: X \times X \rightarrow \mathbb{R}$ is continuous map.

As $X \times X$ is compact
 d is cts.

$d(X \times X)$ is compact hence closed and bounded, hence attains maximum value.

thus, $\exists a, b \in X$ such that

$$\max_{x, y \in X} d(x, y) = d(a, b).$$

claim there is no contraction of X on to X .

Let if possible, F be a on-to contraction map

i.e., $F: X \rightarrow X$ contraction on to map

$\exists a', b' \in X$ such that

$$F(a') = a, \quad F(b') = b.$$

$$d(a, b) = d(F(a'), F(b')) \leq \theta d(a', b') < d(a, b)$$

$$\theta \in [0, 1)$$

where metric on $X \times X$ is

$$d, [(x_1, y_1), (x_2, y_2)] = d(x_1, x_2) + d(y_1, y_2)$$

Continuity of metric space:

$$f: X \times X \rightarrow \mathbb{R}$$

$$f(x, y) = d(x, y)$$

T.S f is cts.

Let $\{(x_n, y_n)\}$ be seq in $X \times X$ s.t. $x_n \rightarrow x$
ie, $y_n \rightarrow y$

$$(x_n, y_n) \rightarrow (x, y)$$

$$\left. \begin{array}{l} x_n \rightarrow x \\ y_n \rightarrow y \end{array} \right\} \Rightarrow \begin{array}{l} d(x_n, x) \rightarrow 0 \\ d(y_n, y) \rightarrow 0 \end{array} \text{ as } n \rightarrow \infty.$$

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y_n) \\ &\leq \underbrace{d(x_n, x)}_{\rightarrow 0} + d(x, y) + \underbrace{d(y, y_n)}_{\rightarrow 0} \end{aligned}$$

$$\underline{d(x_n, y_n) - d(x, y)} \leq \varepsilon \quad \forall \varepsilon > 0$$

$$\begin{aligned} d(x, y) &\leq d(x_n, x) + d(x_n, y) \\ &\leq d(x_n, x) + d(y_n, x) + d(x_n, y_n) \end{aligned}$$

$$\underline{d(x, y) - d(x_n, y_n)} \leq \varepsilon$$

we have that

$$\left| d(x_n, y_n) - d(x, y) \right| \leq \varepsilon$$

hence d is ctd.

Q5 Given an example of a ~~complete~~ ^{Complete} space X and a map $F: X \rightarrow X$ having both of the following properties

a) $\|F_x - F_y\| < \|x - y\|$ whenever $x \neq y$

(b) $F_x \neq x \quad \forall x \in X.$

solⁿ consider $X = [1, \infty)$

$F(x) = x + \frac{1}{x}$ where $F: X \rightarrow X$ mapping

clearly $F_x \neq x \quad \forall x \in [1, \infty).$

$$|F_x - F_y| < |x - y|$$

$$\begin{aligned} \left| x + \frac{1}{x} - y - \frac{1}{y} \right| &= \left| x - y + \frac{y - x}{xy} \right| \\ &= \left| (x - y) \left[1 - \frac{1}{xy} \right] \right| \quad xy^{-1} < xy \\ &< |x - y| \frac{xy}{xy} \end{aligned}$$

$$\Rightarrow \left| x + \frac{1}{x} - y - \frac{1}{y} \right| < |x - y|$$

$$|F_x - F_y| < |x - y|$$

hence it is follows.

Q.7

prove that if X is a compact metric space and F is a mapping from X to X such that $d(Fx, Fy) < d(x, y)$ when $x \neq y$ then F has unique fixed point.

Sol Q.7 (X; d) compact metric space
 $F: X \rightarrow X$ mapping

such that $d(Fx, Fy) < d(x, y)$ $x \neq y$

T.S. F has unique fixed point

Define

$$g(x) = d(Fx, x)$$

hence g is a cts map

being composition of (Fx, x) and d .

$$F: X \rightarrow X \text{ cts}$$

$$x: X \rightarrow X \text{ identity}$$

$$(Fx, x): X \times X \rightarrow X \times X \text{ cts}$$

$$d: X \times X \xrightarrow{\text{cts}} \mathbb{R}$$

As $X \times X$ is compact
 and g is cts. $[g: X \xrightarrow{cts} \mathbb{R}]$
 hence g attains infimum on X .

$\exists x_0 \in X$ such that

$$g(x_0) = \inf_{x \in X} d(Fx, x) = d(Fx_0, x_0).$$

Case (i)

if $g(x_0) = 0$

$$\Rightarrow d(Fx_0, x_0) = 0$$

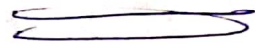
$$Fx_0 = x_0$$

hence x_0 is the fixed point

Case (ii)

if $g(x_0) > 0$

then $d(F^2x_0, Fx_0) < d(Fx_0, x_0) = g(x_0)$



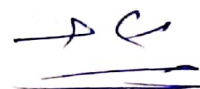
infimum



Uniqueness of fixed point:

Let u, v be two fixed point of F .
 $(u \neq v)$

$$d(u, v) = d(Fu, Fv) < d(u, v)$$



as $Fu = u$ and $Fv = v$ being fixed points, hence the proof follows