DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI

M.Phil./Ph.D. Mathematics Course Work Examination, 2022

MATH21-R07: CONVEX AND NONSMOOTH ANALYSIS

Time: 3 hours | Maximum Marks: 70

Instructions: • Attempt all the questions.

(1) (a) If $A : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, prove that the map $L(x) = A(x) - A(0)$ is linear. If D is a convex set in $\mathbb{R}^m$ prove that $L^{-1}(D)$ is a convex set in $\mathbb{R}^n$. [4 Marks]

(b) State and prove Carathéodory theorem. [4 Marks]

(c) If S is a bounded set in $\mathbb{R}^n$ prove that $\text{cl}(\text{co}S) = \text{co}(\text{cl}S)$. [6 Marks]

(2) (a) Let C_1 and C_2 be two nonempty convex sets in $\mathbb{R}^n$ such that $\text{ri}(C_1 \cap C_2) \neq \emptyset$. Prove that $\text{ri}(C_1 \cap C_2) = \text{ri}C_1 \cap \text{ri}C_2$. [6 Marks]

(b) Find the convex set C in $\mathbb{R}^2$ where $A(C) = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 + x_2 \leq 4, -x_1 + x_2 \leq 4, x_2 \geq 3\}$ and the affine mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $A(x_1, x_2) = (x_1 + 1, x_1 + x_2)$. [4 Marks]

(c) If $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a positive definite symmetric linear operator any x with $\langle Qx, x\rangle = 1$ is an extreme point of the convex set $C = \{x \in \mathbb{R}^n : \langle Qx, x\rangle \leq 1\}$. [4 Marks]

$\|x\|_2 = \sqrt{x_1^2+x_2^2}$

(3) (a) Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Prove that a point $y_x \in C$ is the ~~projection $p_C(x)$ if and only if~~ [6 Marks]

$$\langle x - y_x, y - y_x\rangle \leq 0, \forall y \in C.$$

Find the projection of $(2, 1)$ onto $C = \{x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$.

(b) Prove that the tangent cone to a closed convex set $C \subseteq \mathbb{R}^n$ at $x \in C$ is the closure of the cone generated by $C - x$. $T_C(x) = \text{cone}(C-x) = cl(\mathbb{R}^+(C-x))$ [4 Marks]

(c) Let C_1 and C_2 be two nonempty closed convex sets in $\mathbb{R}^n$. For $x \in C_1 \cap C_2$ prove that $N_{C_1}(x) + N_{C_2}(x) \subseteq N_{C_1 \cap C_2}(x)$. [4 Marks]

(PPT-21 majhar)

(4) (a) Let $f_1, f_2 \in \text{Conv}\mathbb{R}^n$ such that they have a common minorant. Prove that their infimal convolution is also in $\text{Conv}\mathbb{R}^n$. [6 Marks]

Lipschitz condition

(b) Let $f \in \text{Conv}\mathbb{R}^n$ and $S \subseteq \text{ri}(\text{dom}f)$ be convex and compact. Prove that there exists $L \geq 0$ such that [8 Marks]

$$|f(x) - f(x')| \leq L\|x - x'\|, \forall x, x' \in S.$$

$\|x\|^4$, $x=(x_1,x_2)$

(5) (a) Give an example of a strictly convex function defined on $\mathbb{R}^2$ which is not strongly convex. [2 Marks]

(PPT-21 Rohit)

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Prove that a vector $s \in \mathbb{R}^n$ is a subgradient of f at x if and only if $(s, -1) \in \mathbb{R}^n \times \mathbb{R}$ is normal to epi$f$ at $(x, f(x))$. Illustarte this theorem geometrically and analytically for the function $f(x, y) = |x| + |y|$ at $((x, y), f(x, y))$ for $(x, y) = (0, 0)$. [6 Marks]

(PPT-21 ashish yadav)

(c) Prove that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex with modulus $c > 0$ on a convex set C if and only if for all $x_1, x_2 \in C$, [6 Marks]

$$\langle s_2 - s_1, x_2 - x_1\rangle \geq c\|x_2 - x_1\|^2, \forall s_i \in \partial f(x_i), i = 1, 2.$$

$(a+b)^3 = a^3+b^3$

$A(\alpha x + (1-\alpha)x) - A(0)$

$\alpha x - \alpha 0$, $(1-\alpha)0$